MATH-UA 123 Calculus 3: Directional Derivatives, Gradient, Contours, Maximum and Minimum Values of a Function

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START RECORDING
LIVE TRANSCRIPT

REMINDER

First Midterm
is on Monday, October 25

Directional Derivative of a Function

- Consider a function f on a domain D in $\mathbb{R}^2$ or $\mathbb{R}^3$
- Given a point $\vec{r}_0$ in the domain and a vector $\vec{v}_0$, the directional derivative of f in the direction $\vec{v}_0$ at $\vec{r}_0$ is defined to be

$$D_{\vec{v}} f(\vec{r}_0) = \left.\frac{d}{dt}\right|_{t=0} f(\vec{r}(t))$$

- Define the gradient of f to be the vector field

$$\vec{\nabla} f = \langle f_x, f_y, f_z \rangle$$

- The chain rule shows that

$$D_{\vec{v}} f(\vec{r}) = \vec{v} \cdot \vec{\nabla} f(\vec{r})$$

Examples of Directional Derivatives

- The directional derivative of a linear function $f(x,y,z) = ax + by + cz + d$ in a direction $\vec{u}$ is

$$D_{\vec{u}}f(x,y,z) = \vec{u} \cdot \langle f_x, f_y, f_z \rangle = \vec{u} \cdot \langle a, b, c \rangle,$$

  which is constant for all points (x,y,z)
- The directional derivative of the function

$$f(x,y,z) = x^2 + y^2 + z^2 = |\vec{r}|^2$$

  in a direction $\vec{u}$ is

$$\begin{aligned} D_{\vec{u}}f(x,y,z) = \vec{u} \cdot \langle f_x, f_y, f_z \rangle &= \vec{u} \cdot \langle 2x, 2y, 2z \rangle \\ &= 2\vec{u} \cdot \langle x, y, z \rangle \\ &= 2\vec{u} \cdot \vec{r} \end{aligned}$$

Vector Fields and the Gradient of a Function

$$\vec{r} = \langle x, y, z \rangle \longrightarrow \boxed{\vec{V}} \longrightarrow \vec{V}(\vec{r}) = \langle V_1(x, y, z), V_2(x, y, z), V_3(x, y, z) \rangle$$

- A vector field is a function where the input is a point in space and the output is a vector
- The gradient of a scalar function of space is a vector field

$$\vec{\nabla} f(\vec{r}) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

$$\vec{r} = \langle x, y, z \rangle \longrightarrow \boxed{\vec{\nabla} f} \longrightarrow \vec{\nabla} f(\vec{r}) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

Directional Derivatives and the Gradient

- The directional derivative of a function f at a point $\vec{r}$ in the direction $\vec{u}$ is defined to be

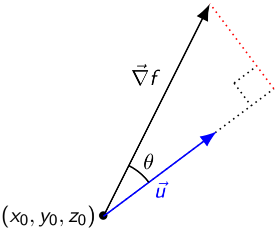
$$D_{\vec{u}} f(\vec{r}) = \vec{u} \cdot \vec{\nabla} f(\vec{r})$$

- Since $\vec{u}$ is a unit vector,

$$D_{\vec{u}} f(\vec{r}) = |\vec{u}||\vec{\nabla} f(\vec{u})| \cos\theta = |\vec{\nabla} f(\vec{u})| \cos\theta,$$

  where θ is the angle between the direction $\vec{u}$ and the vector $\vec{\nabla} f(\vec{r})$
- The directional derivative of f at $\vec{r}$ is greatest when $\vec{u}$ points in the same direction as $\vec{\nabla} f(\vec{r})$
- The directional derivative of f at $\vec{r}$ is most negative when $\vec{u}$ points in the opposite direction to $\vec{\nabla} f(\vec{r})$
- The directional derivative of f at $\vec{r}$ is zero when $\vec{u}$ is orthogonal to $\vec{\nabla} f(\vec{r})$

Directional Derivatives and the Gradient



- The directional derivative of f in the direction $\vec{u}$ at $\vec{r}$ is

$$D_{\vec{u}} f(\vec{r}) = \vec{u} \cdot \vec{\nabla} f(\vec{r})$$

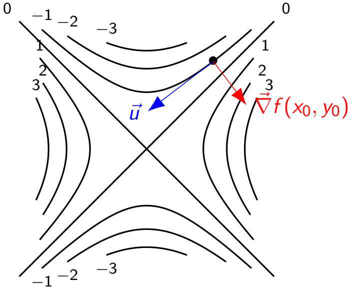
- Since $\vec{u}$ is a unit vector,

$$D_{\vec{u}} f(\vec{r}) = |\vec{u}||\vec{\nabla} f(\vec{r})| \cos\theta = |\vec{\nabla} f(\vec{r})| \cos\theta,$$

  where θ is the angle between $\vec{u}$ and $\vec{\nabla} f(\vec{r})$

- $$-|\vec{\nabla} f(\vec{r})| \leq D_{\vec{u}} f(\vec{r}) \leq |\vec{\nabla} f(\vec{r})|$$

The Gradient is Orthogonal to Each Level Set



- Since f is constant along a contour,

$$D_{\vec{u}} f(\vec{r}) = 0$$

  if $\vec{u}$ is a direction tangent to the contour at $\vec{r}$
- Therefore, $\vec{u} \cdot \vec{\nabla} f(\vec{r}) = 0$ for any direction $\vec{u}$ tangent to the contour at $\vec{r}$
- Therefore, $\vec{\nabla} f(\vec{r})$ is normal to the contour at $\vec{r}$

The Gradient is Orthogonal to Each Level Set

- Consider the contour $f = c$ in 3-space
- Suppose $f(x_0, y_0, z_0) = c$ and $\vec{v}$ is tangent to the contour at (x_0, y_0, z_0)
- Let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ be a parameterized curve lying in the contour such that $\vec{r}(0) = \langle x_0, y_0, z_0 \rangle$ and $\vec{r}'(0) = \vec{v}$
- Since the curve lies in the contour $f = c$,

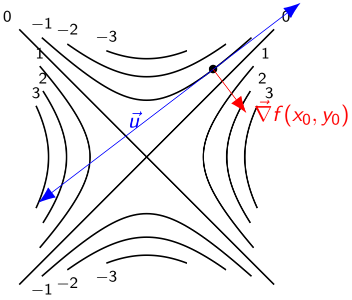
$$f(x(t), y(t), z(t)) = c, \text{ for all } t \text{ in the domain of } \vec{r}$$

- Differentiating this using the chain rule, we get

$$\begin{aligned} 0 &= \frac{d}{dt} f(x(t), y(t), z(t)) \\ &= f_x(x(t), y(t), z(t)) x'(t) + f_y(x(t), y(t), z(t)) y'(t) \\ &\quad + f_z(x(t), y(t), z(t)) z'(t) \\ &= \vec{r}'(t) \cdot \vec{\nabla} f(\vec{r}(t)) \end{aligned}$$

- At $t = 0$, we get $\vec{v} \cdot \vec{\nabla} f(x_0, y_0, z_0) = 0$

Tangent Line to Level Set in 2-Space



- Consider a level set $f(x, y) = c$ and a point (x_0, y_0) in the contour
- The line tangent to the level set at (x_0, y_0) passes through (x_0, y_0) and is orthogonal to $\vec{\nabla} f$
- Given any $(x, y,)$ in the tangent line, the vector $\langle x - x_0, y - y_0 \rangle$ is parallel to the line
- Therefore, the equation of the line is

$$\vec{n} \cdot \langle x - x_0, y - y_0 \rangle = 0, \text{ where } \vec{n} = \vec{\nabla} f(x_0, y_0)$$

Example of Line Tangent to Contour in 2-Space

- Suppose $g(x, y) = x^2 - y^2$ and we want the tangent line to the contour passing through $(2, 1)$
- Contour passing through $(2, 1)$ is $g(x, y) = 3$
- Gradient of g is $\vec{\nabla} g = \langle 2x, -2y \rangle$
- Gradient of g at $(2, 1)$ is $\vec{\nabla} g(2, 1) = \langle 4, -2 \rangle$
- Equation of line passing through (x_0, y_0) and normal to $\vec{n}$ is

$$\vec{n} \cdot \langle x - x_0, y - y_0 \rangle = 0$$

- The equation of the tangent line is

$$\begin{aligned} \vec{\nabla} g(2, 1) \cdot \langle x - 2, y - 1 \rangle &= 0 \\ \langle 4, -2 \rangle \cdot \langle x - 2, y - 1 \rangle &= 0 \\ 4(x - 2) - 2(y - 1) &= 0 \\ 4x - 2y &= 6 \end{aligned}$$

Example of Line Tangent to Contour in 2-Space

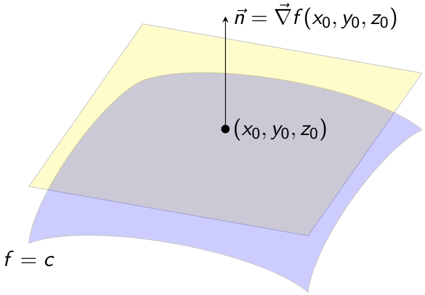
- Suppose $f(1,3) = -2,\ f_x(1,3) = 4,\ f_y(1,3) = -17$
- Find equation of plane tangent to contour $f = -2$ at $(1,3)$
- No formula for $\vec{\nabla} f(x,y)$
- $\vec{\nabla} f(1,3) = \langle 4, -17 \rangle$
- Equation of line passing through (x_0, y_0) and normal to $\vec{n}$ is

$$\vec{n} \cdot \langle x - x_0, y - y_0 \rangle = 0$$

- The equation of the tangent line is

$$\begin{aligned}
\vec{\nabla} f(1,3) \cdot \langle x - 1, y - 3 \rangle &= 0 \\
\langle 4, -17 \rangle \cdot \langle x - 1, y - 3 \rangle &= 0 \\
4(x-1) - 17(y-3) &= 0 \\
4x - 17y &= -47
\end{aligned}$$

Plane Tangent to Level Set in 3-Space



- Suppose (x_0, y_0, z_0) lies in the contour $f(x, y, z) = c$
- The gradient of f is normal to the plane tangent to the contour at (x_0, y_0, z_0)
- If a plane contains a point (x_0, y_0, z_0) and has a normal $\vec{n}$, then an equation for it is

$$\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

- Here, we can set $\vec{n} = \vec{\nabla} f(x_0, y_0, z_0)$

Example of Plane Tangent to Contour in 3-Space

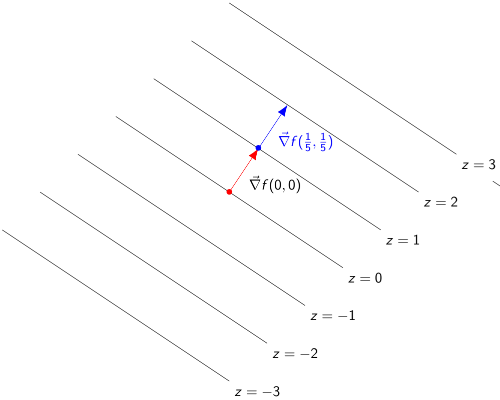
- Suppose $f(1,2,3) = -5$, $f_x(1,2,3) = 7$, $f_y(1,2,3) = -11$, $f_z(1,2,3) = 13$
- Find equation of plane tangent to contour $f = -5$ at $(1,2,3)$
- No formula for $\vec{\nabla} f(x,y)$
- $\vec{\nabla} f(1,2,3) = \langle 7, -11, 13 \rangle$
- Equation of plane with normal $\vec{n}$ and containing (x_0, y_0, z_0) is

$$\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

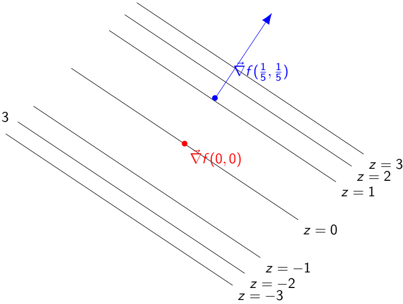
- The equation of the tangent plane is

$$\begin{aligned} \vec{\nabla} f(1,2,3) \cdot \langle x-1, y-2, z-3 \rangle &= 0 \\ \langle 7, -11, 13 \rangle \cdot \langle x-1, y-2, z-3 \rangle &= 0 \\ 7(x-1) - 11(y-2) + 13(z-3) &= 0 \\ 7x - 11y + 13z &= 24 \end{aligned}$$

Contours of Plane Versus Parabolic Cylinder

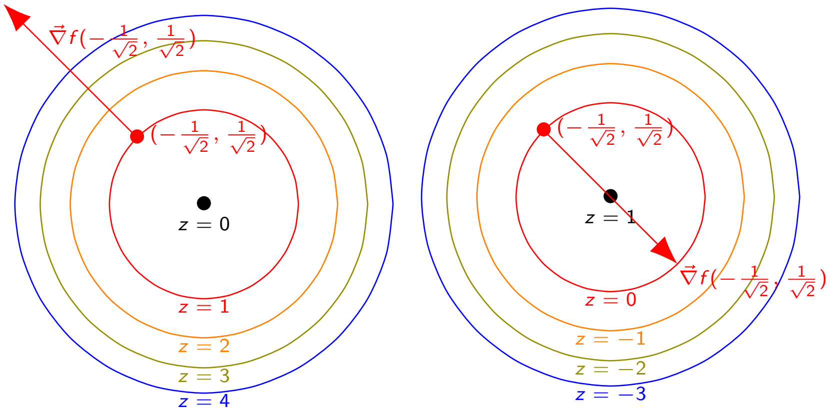


$$z = 2x + 3y$$



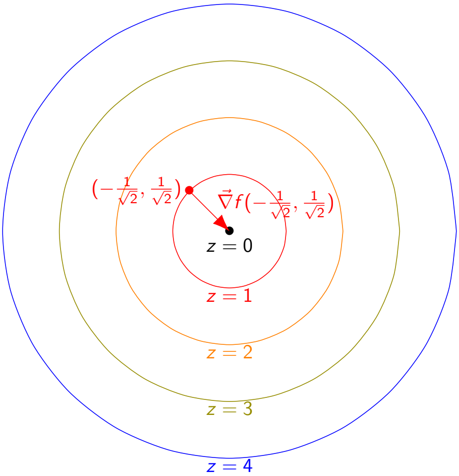
$$z = (2x + 3y)^2$$

Contours of Circular Paraboloids



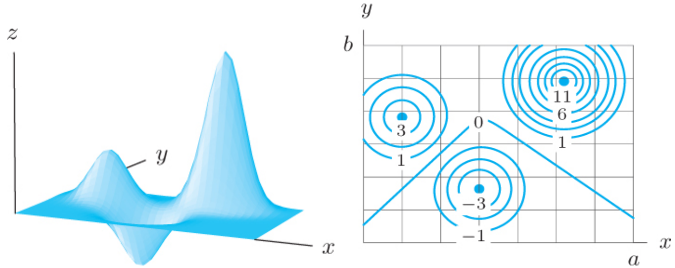
$z = x^2 + y^2$ $\qquad$ $z = 1 - x^2 - y^2$

Contour of Circular Cone



$$z = \sqrt{x^2 + y^2}$$

Shape Versus Contours of a Graph

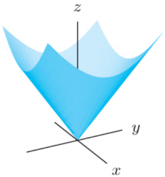


- Key features of a surface:
  - Peaks
  - Bottoms
  - Ridges and valleys between peaks or bottoms
- There are at least 4 points where the gradient is zero
  - Two peaks
  - One bottom
  - One point in between the peaks and bottom, where the contour consists of two intersecting curves

Some Possible Shapes of a Graph

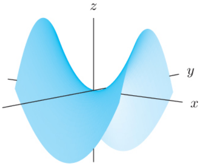
Isolated local maximum

Isolated local minimum

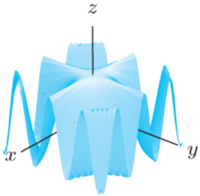


Isolated local minimum

Line of local minima



Saddle point

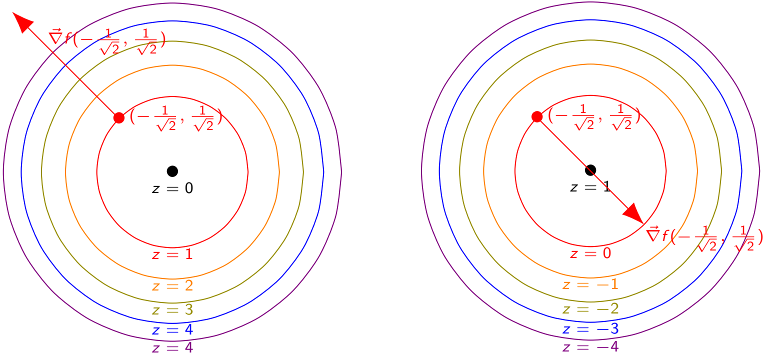


Complicated set of local maxima

Critical Point of Function

- A point (x_0, y_0) is a critical point of a function f if
  - It is in the domain of f
  - The gradient $\vec{\nabla} f(x_0, y_0, z_0)$ is either zero or undefined
- Possible shapes of a surface near a critical point
  - Isolated local maximum: Top of a hill
  - Isolated local minimum: Bottom of a bowl
  - Curve of local minima: Bottom of a valley
  - Curve of local maxima: Top of a ridge
  - Saddle point
  - Other

Contours of Circular Paraboloids



$$f(x, y) = x^2 + y^2$$

$$\vec{\nabla} f(x, y) = 2\langle x, y\rangle$$

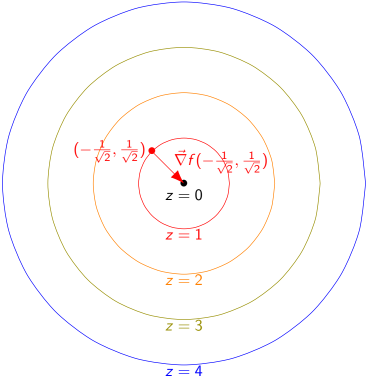
Local minimum

$$f(x, y) = 1 - x^2 - y^2$$

$$\vec{\nabla} f(x, y) = -2\langle x, y\rangle$$

Local maximum

Contour of Circular Cone

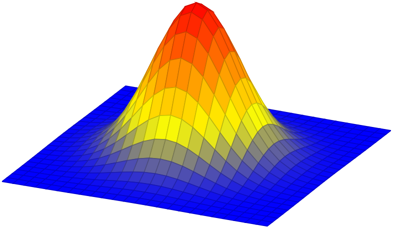


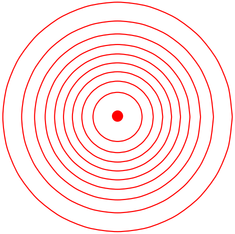
$$f(x, y) = \sqrt{x^2 + y^2}$$

$$\vec{\nabla} f = \frac{\langle x, y \rangle}{\sqrt{x^2 + y^2}}$$

Local minimum

Another Example of Local Maximum

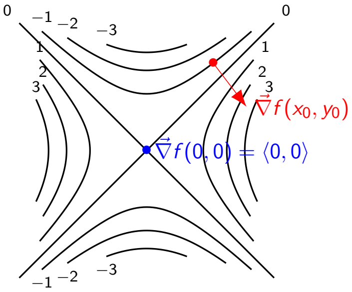




$$x^2 + y^2 = -\ln(c)$$

$$z = e^{-x^2-y^2}$$

Saddle Point



- A point (x_0, y_0) is a saddle point of the graph $z = f(x, y)$, if the contours of f near (x_0, y_0) look like the above
- Examples
  - $(0, 0)$ for the function $f(x, y) = x^2 - y^2$
  - $(0, 0)$ for the function $f(x, y) = xy$

Critical Point of Function

- A critical point of a function f is a point in the domain of f where $\vec{\nabla} f$ is either undefined or equal to the zero vector
- Examples
  - $f(x, y) = \sqrt{x^2 + y^2}$: $\vec{\nabla} f(x, y) = \left\langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right\rangle$ is undefined at $(0, 0)$
  - $f(x, y) = e^{-x^2-y^2}$: $\vec{\nabla} f(x, y) = -2e^{-x^2-y^2}\langle x, y, \rangle$ is the zero vector at $(0, 0)$
  - $f(x, y) = x^2 - y^2$: $\vec{\nabla} f(x, y) = 2\langle x, -y, \rangle$ is the zero vector at $(0, 0)$
- If $\vec{\nabla} f(x_0, y_0) = \langle 0, 0 \rangle$, then the tangent plane

  $$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = f(x_0, y_0)$$

  is horizontal

Types of Critical Points

- Suppose (x_0, y_0) is a critical point of a function $f(x, y)$
- A point (x_0, y_0) is a **local maximum**, if

$$f(x, y) \leq f(x_0, y_0) \text{ for all } (x, y) \text{ near } (x_0, y_0)$$

  - Example: $(0, 0)$ for $f(x, y) = -x^2 - y^2$
- A point (x_0, y_0) is a **local minimum**, if

$$f(x, y) \geq f(x_0, y_0) \text{ for all } (x, y) \text{ near } (x_0, y_0)$$

  - Example: $(0, 0)$ for $f(x, y) = x^2 + y^2$
- A point (x_0, y_0) is a **saddle point**, if it meets the criteria in previous slide
  - Example: $(0, 0)$ for $f(x, y) = x^2 - y^2$
- There are other types of critical points that we will not study
  - Example: $(0, 0)$ for $f(x, y) = xy(x^2 - y^2)$

Tests for Critical Point Type

- Analyze formula of function
- Draw graph
- Draw contours
- Second derivative test