MATH-UA 123 Calculus 3: Lineaar Approximation, Tangent Plane, Differentials, Chain Rule

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LIVE TRANSCRIPT

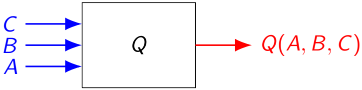
REMINDER

Next week's Monday lecture is on TUESDAY, October 12

REMINDER

First Midterm
is on Monday, October 25

Partial Derivatives



- Function: Q
  - Input values: A, B, C
  - Output value: $Q(A, B, C)$
- Partial derivative of Q with respect to B:

$$Q_B = \frac{\partial Q}{\partial B} \simeq \frac{\text{Change in Q}}{\text{Small change in B}}$$

  with A and C assumed to be constants
- Change in Q due to small change in B

$$\text{Change in } Q \simeq Q_B(\text{Change in } B)$$

- Change in Q due to small changes in A, B, C

$$\text{Change in } Q \simeq Q_A(\text{Change in } A) + Q_B(\text{Change in } B) + Q_C(\text{Change in } C)$$

Example of Linear Approximation

- Suppose $f(x,y)$ is a function such that

$$f(1,-1) = 2,\ f_x(1,-1) = -3,\ f_y(1,-1) = 1$$

- Since

$$\text{Change in } f \simeq f_x(\text{small change in } x + f_y(\text{small change in } y,$$

it follows that

$$\begin{aligned} & f(1.1,-1.2) - f(1,-1) \\ & \simeq f_x(1,-1)(1.1-1) + f_y(1,-1)(-1.2-(-1.1)) \end{aligned}$$

- Equivalently,

$$\begin{aligned} & f(1.1,-1.2) \\ & \simeq f(1,-1) + f_x(1,-1)(1.1-1) + f_y(1,-1)(-1.2-(-1.1)) \\ & = 2 + (-3)(0.1) + 1(-0.2) \\ & = 2 - 0.3 - 0.2 = 1.5 \end{aligned}$$

Tangent Plane

- Suppose $f(x, y)$ is a function such that

$$f(1, -1) = 2, \; f_x(1, -1) = -3, \; f_y(1, -1) = 1$$

- Linear Approximation: If (x, y) is clos to $(1, -1)$, then

$$\begin{aligned} f(x, y) &\simeq f(1, -1) + f_x(1, -1)(x - 1) + f_y(1, -1)(y - (-1)) \\ &= 2 - 3(x - 1) + 1(y + 1) \\ &= -3x + y + 4 \end{aligned}$$

- The graph

$$z = -3x + y + 4$$

  is called the tangent plane of f at $(1, -1)$
- The tangent plane of f at $(1, -1)$ touches the graph of f at $(1, -1)$

Linear Approximation

- Change in $f(x, y)$ due to small changes in x, y

$$\text{Change in } f \simeq f_x(\text{Change in } x) + f_y(\text{Change in } y)$$

- If (x, y) is close to (x_0, y_0), then

$$f(x, y) \simeq f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

- If x_0, y_0 are viewed as constants, then this is the same as

$$f(x, y) \simeq ax + by + c,$$

  where

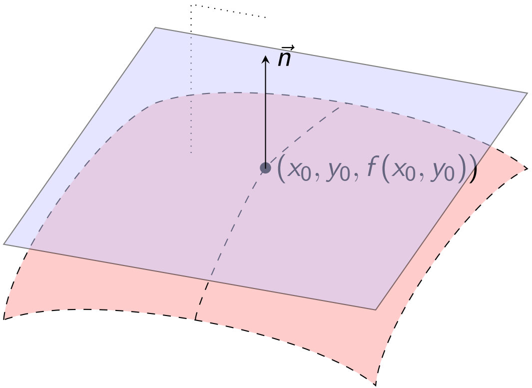
$$a = f_x(x_0, y_0), \;\; b = f_y(x_0, y_0),$$
$$c = f(x_0, y_0) - x_0 f_x(x_0, y_0) - y_0 f_y(x_0, y_0)$$

- The graph of the linear approximation is a plane

$$z = ax + by + c = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

- This is called the tangent plane of f at (x_0, y_0)

Tangent Plane



Differential of a Function

- Recall that if $f(x)$ is a function with one input, then

$$df = \frac{df}{dx}\,dx$$

- The differential of a function $A(P, Q, R)$ is

$$dA = \frac{\partial A}{\partial P}\,dP + \frac{\partial A}{\partial Q}\,dQ + \frac{\partial A}{\partial R}\,dR = A_P\,dP + A_Q\,dQ + A_R\,dR$$

- The differentials dP, dQ, dR are small changes in inputs
- dA is the resulting change in output

Rules of Differentials

- Sum rule: $d(f+g) = df + dg$
- Constant factor rule: $d(cf) = c\,df$
- Product rule: $d(fg) = g\,df + f\,dg$
- Quotient rule: $d\left(\frac{f}{g}\right) = \frac{g\,df - f\,dg}{g^2}$
- Example: Suppose $f(x,y,z) = \frac{ye^z}{x}$

$$\begin{aligned} df &= \frac{x\,d(ye^z) - (ye^z)\,dx}{x^2} \\ &= \frac{x(e^z\,dy + ye^z\,dz) - ye^z\,dx}{x^2} \\ &= \frac{-ye^z\,dx + xe^z\,dy + ye^z\,dz}{x^2} \end{aligned}$$

Linear Approximation Using Differential

- Suppose (x, y) is close to (x_0, y_0)
- $f(x_0, y_0,\ f_x(x_0, y_0),\ f_y(x_0, y_0),\ f_z(x_0, y_0)$ are known
- Estimate $f(x, y)$
- Differential of f:

$$df = f_x\, dx + f_y\, dy$$

- Differentials represent small changes:

$$\begin{aligned} dx &\simeq x - x_0 \\ dy &\simeq y - y_0 \\ df &\simeq f(x, y) - f(x_0, y_0) \end{aligned}$$

- Change in f is therefore

$$f(x, y) - f(x_0, y_0) \simeq f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example of Linear Approximation Using Differential

- Suppose

$$
\begin{aligned}
f(1,2) &= 2 \\
f_x(1,2) &= 3 \\
f_y(1,2) &= 5
\end{aligned}
$$

- Suppose we want to estimate $f(0.8, 2.1)$
- View each differential as a small change:

$$
\begin{aligned}
dx &\simeq 0.8 - 1 = -0.2 \\
dy &\simeq 2.1 - 2 = 0.1 \\
df &\simeq f(0.8, 2.1) - f(1,2)
\end{aligned}
$$

- Therefore,

$$
\begin{aligned}
df &= f_x\, dx + f_y\, dy \\
f(0.8, 2.1) - f(1,2) &\simeq 3(0.8 - 1) + 5(2.1 - 2) \\
f(0.8, 2.1) &\simeq 2 - 0.6 + 0.5 \\
&= 1.9
\end{aligned}
$$

Implicit Differentiation Using Differentials

- Suppose we want to find the partial derivatives of $f(x, y)$, where the graph $z = f(x, y)$ satisfies

$$x^2 + y^2 + z^2 = 3xyz$$

- Take differential of both sides of equation

$$2x\,dx + 2y\,dy + 2z\,dz = 3yz\,dx + 3xz\,dy + 3xy\,dz$$

- Solve for dz:

$$\begin{aligned}(2z - 3xy)\,dz &= (3yz - 2x)\,dx + (3xz - 2y)\,dy \\ dz &= \frac{(3yz - 2x)\,dx + (3xz - 2y)\,dy}{2z - 3xy}\end{aligned}$$

- Since $z = f(x, y)$, $dz = f_x\,dx + f_y\,dy$
- Therefore,

$$f_x = \frac{3yz - 2x}{2z - 3xy} \text{ and } f_y = \frac{3xz - 2y}{2z - 3xy}$$

Ideal Gas Law

- The ideal gas law says

$$\frac{PV}{T} = k,$$

where k is a physical constant and

$$P = \text{pressure},\ V = \text{volume},\ T = \text{temperature}$$

- Any variable can be viewed as a function of the other two
- Suppose we want formulas for P_V, P_T, T_V, T_P, V_P, V_T
- Rewrite equation: $PV = kT$
- Compute differential: $V\,dP + P\,dV = k\,dT$
- To compute P_V and P_T, solve for dP:

$$dP = -\frac{P}{V}\,dV + \frac{k}{V}\,dT$$

- Therefore, $P_V = -\frac{P}{V}$ and $P_T = \frac{k}{V}$

Tangent Plane Using Differentials

- Suppose we want to find the tangent plane of

$$x^2 + y^2 + z^2 = 3xyz$$

  at $(1, 1, 2)$

- Differential:

$$2x\,dx + 2y\,dy + 2z\,dz = 3yz\,dx + 3xz\,dy + 3xy\,dz$$

- Only $(x, y, z) = (1, 1, 2)$ matters:

$$\begin{aligned} 2\,dx + 2\,dy + 4\,dz &= 6\,dx + +6\,dy + 3\,dz \\ dz &= 4\,dx + 4\,dy \\ z - 2 &= 4(x-1) + 4(y-1) \end{aligned}$$

- The tangent plane is therefore given by

$$z = 4x + 4y - 6$$

Chain Rule Using Differentials

- Suppose $f(x, y) = xe^{-y}$
- Define a new function $h(p) = f(1 + 2p, 1 - 3p)$
- We want to compute $h'(p)$
- On one hand, $dh = h'(p)\,dp$
- On the other, $h(p) = f(x, y)$, where $x = 1 + 2p$ and $y = 1 - 3p$
- Therefore,

$$\begin{aligned}
dx &= 2\,dp \\
dy &= -3\,dp \\
dh &= df \\
&= e^{-y}\,dx - xe^{-y}\,dy \\
&= e^{3p-1}2\,dp - (1 + 2p)e^{3p-1}(-3)\,dp \\
&= e^{3p-1}(2 + (3 + 6p))\,dp \\
&= e^{3p-1}(5 + 6p)\,dp
\end{aligned}$$

- Therefore, $h'(p) = e^{3p-1}(5 + 6p)$

Chain Rule Using Differentials

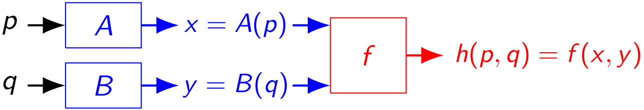
- Suppose
$$f(x, y) = (2y + 3)^2 e^{5x-4}$$
and we want to calculate f_x and f_y
- Write this as $f = p^2 e^q$, where $p = 2y + 3$ and $q = 5x - 4$
- Then

$$\begin{aligned}
dp &= 2\,dy \\
dq &= 5\,ddx \\
df &= 2pe^q\,dp + p^2 e^q\,dq \\
&= 2(2y + 3)e^{5x-4}\,dx + (2y + 3)^2 e^{5x-4}\,dy
\end{aligned}$$

- Since $df = f_x\,dx + f_y\,dy$,

$$\begin{aligned}
f_x &= 2(2y + 3)e^{5x-4} \\
f_y &= (2y + 3)^2 e^{5x-4}
\end{aligned}$$

Chain Rule for Functions with Multiple Inputs



- Suppose we have a function $f(x, y)$ and functions $A(p)$ and $B(p)$
- Define a new function $h(p) = f(A(p), B(p))$
- Two step description:
  - Given p, set $x = A(p)$ and $y = B(p)$
  - Set $h(p) = f(x, y)$
- We want to compute the sensitivity of h to a small change in p, i.e., $h'(p)$

Chain Rule Using Differentials

- $h(p) = f(A(p), B(p)) = f(x, y)$, where

$$x = A(p) \text{ and } y = B(p)$$

- A small change dp causes
  - A small change $dx = A'(p)\,dp$
  - A small change $dy = B'(p)\,dp$
- The small changes dx and dy in turn causes a small change in f, which is also the resulting change in f

$$\begin{aligned} dh &= d(f(x, y)) \\ &= f_x(x, y)\,dx + f_y(x, y)\,dy \\ &= f_x(A(p), B(p))A'(p)\,dp + f_y(A(p), B(p))B'(p)\,dp \\ &= (f_x(A(p), B(p))A'(p) + f_y(A(p), B(p))B'(p))\,dp \end{aligned}$$

- Since $dh = h'(p)\,dp$,

$$h'(p) = f_x(A(p), B(p))A'(p) + f_y(A(p), B(p))B'(p)$$

Chain Rule Using Differentials

- Suppose $f(x, y) = xe^y$
- Define a new function
  $h(p, q) = f(p + q, p - q) = (p + q)^2 e^{p-q}$
- We want to compute the partial derivatives $h_p(p, q)$ and $h_q(p, q)$
- If $x = p + q$ and $y = p - q$, then $h(p, q) = f(x, y)$
- Their differentials are $dx = dp + dq$ and $dy = dp - dq$
- Since $h(p) = f(x, y) = xe^y$,

$$\begin{aligned} dh &= df \\ &= e^y\,dx + xe^y\,dy \\ &= e^{p-q}\,(dp + dq) + (p + q)e^{p-q}\,(dp - dq) \\ &= e^{p-q}(1 + p + q)\,dp + e^{p-q}(1 - p - q)\,dq \end{aligned}$$

- Since $dh = h_p\,dp + h_q\,dq$,

$$\begin{aligned} h_p(p, q) &= e^{p-q}(1 + p + q) \\ h_q(p, q) &= e^{p-q}(1 - p - q) \end{aligned}$$

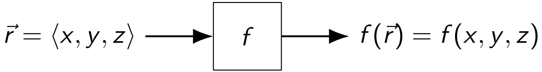
Doing the Chain Rule Directly

- If $h(p, q) = (p + q)e^{p=q}$, then

$$
\begin{aligned}
h_p(p, q) &= \partial_p(p + q)e^{p-q} + (p + q)\partial_p(e^{p-q}) \text{ by the product rule} \\
&= e^{p-q} + (p + q)e^{p-q} \\
&= (1 + p + q)e^{p-q} \\
h_q(p, q) &= \partial_q(p + q)e^{p-q} + (p + q)\partial_q(e^{p-q}) \text{ by the product rule} \\
&= e^{p-q} - (p + q)e^{p-q} \\
&= (1 - p + q)e^{p-q}
\end{aligned}
$$

- Two separate calculations required, one for each partial derivative
- Using differentials requires only one calculation

Derivative of a Function Along a Curve



- Suppose f is a function of points in space
- Suppose $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a parameterized curve such that $\vec{r}(0) = \vec{r}_0 = \langle x_0, y_0, z_0\rangle$
- The derivative of f along the curve $\vec{r}(t)$ is defined to be

$$\begin{aligned}\frac{d}{dt} f(\vec{r}(t)) &= \frac{d}{dt} f(x(t), y(t), z(t)) \\ &= f_x(\vec{r}(t))x'(t) + f_y(\vec{r}(t))y'(t) + f_z(\vec{r}(t))z'(t)\end{aligned}$$

- In particular, when $t = 0$,

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} f(\vec{r}(t)) &= f_x(\vec{r}_0)x'(0) + f_y(\vec{r}_0)y'(0) + f_z(\vec{r}_0)z'(0) \\ &= \langle f_x(\vec{r}_0), f_y(\vec{r}_0), f_z(\vec{r}_0)\rangle \cdot \langle x'(0), y'(0), z'(0)\rangle \\ &= \vec{\nabla} f(\vec{r}_0) \cdot \vec{r}'(0),\end{aligned}$$

where $\vec{\nabla} f = \langle f_x, f_y, f_z\rangle$

Directional Derivative of a Function

- Given point $\vec{r}_0$, any vector $\vec{v}_0$, and any curve $\vec{r}(t)$ such that $\vec{r}(0) = \vec{r}_0$ and $\vec{r}'(0) = \vec{v}_0$,

$$\left.\frac{d}{dt}\right|_{t=0} f(\vec{r}(t)) = \vec{\nabla} f(\vec{r}_0) \cdot \vec{r}'(0) = \vec{v}_0 \cdot \vec{\nabla} f(\vec{r}_0)$$

- The value of the right side depends only on f, $\vec{r}_0$, and $\vec{v}_0$ and not on the curve
- Define the gradient of f to be the vector field

$$\vec{\nabla} f = \langle f_x, f_y, f_z \rangle$$

- Given a function f, a vector $\vec{v}$, and a position $\vec{r}$, denote

$$D_{\vec{v}} f(\vec{r}) = \vec{v} \cdot \vec{\nabla} f(\vec{r})$$

- The directional derivative of f at $\vec{r}$ in the direction of the vector $\vec{v}$ is

$$D_{\vec{u}} f(\vec{r}) = \vec{u} \cdot \vec{\nabla} f(\vec{r}),$$

where $\vec{u} = \dfrac{\vec{v}}{|\vec{v}|}$ is the direction of $\vec{v}$

Examples of Directional Derivatives

- The directional derivative of a linear function $f(x, y, z) = ax + by + cz + d$ in a direction $\vec{u}$ is

$$D_{\vec{u}} f(x, y, z) = \vec{u} \cdot \langle f_x, f_y, f_z \rangle = \vec{u} \cdot \langle a, b, c \rangle,$$

  which is constant for all points (x, y, z)
- The directional derivative of the function

$$f(x, y, z) = x^2 + y^2 + z^2 = |\vec{r}|^2$$

  in a direction $\vec{u}$ is

$$\begin{aligned} D_{\vec{u}} f(x, y, z) = \vec{u} \cdot \langle f_x, f_y, f_z \rangle &= \vec{u} \cdot \langle 2x, 2y, 2z \rangle \\ &= 2\vec{u} \cdot \langle x, y, z \rangle \\ &= 2\vec{u} \cdot \vec{r} \end{aligned}$$

Vector Fields and the Gradient of a Function

$$\vec{r} = \langle x, y, z \rangle \longrightarrow \boxed{\vec{V}} \longrightarrow \vec{V}(\vec{r}) = \langle V_1(x,y,z), V_2(x,y,z), V_3(x,y,z) \rangle$$

- A vector field is a function where the input is a point in space and the output is a vector
- The gradient of a scalar function of space is a vector field

$$\vec{\nabla} f(\vec{r}) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$$

$$\vec{r} = \langle x, y, z \rangle \longrightarrow \boxed{\vec{\nabla} f} \longrightarrow \vec{\nabla} f(\vec{r}) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$$

Directional Derivatives and the Gradient

- The directional derivative of a function f at a point $\vec{r}$ in the direction $\vec{u}$ is defined to be

$$D_{\vec{u}}f(\vec{r}) = \vec{u} \cdot \vec{\nabla} f(\vec{u})$$

- Since $\vec{u}$ is a unit vector,

$$D_{\vec{u}}f(\vec{r}) = |\vec{u}||\vec{\nabla} f(\vec{u})| \cos\theta = |\vec{\nabla} f(\vec{u})| \cos\theta,$$

  where θ is the angle between the direction $\vec{u}$ and the vector $\vec{\nabla} f(\vec{r})$
- The directional derivative of f at $\vec{r}$ is greatest when $\vec{u}$ points in the same direction as $\vec{\nabla} f(\vec{r})$
- The directional derivative of f at $\vec{r}$ is most negative when $\vec{u}$ points in the opposite direction to $\vec{\nabla} f(\vec{r})$
- The directional derivative of f at $\vec{r}$ is zero when $\vec{u}$ is orthogonal to $\vec{\nabla} f(\vec{r})$