MATH-UA 123 Calculus 3: Limits, Continuity, Partial Derivatives, Linear Approximations

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LIVE TRANSCRIPT

REMINDER

Next week's Monday lecture is on TUESDAY, October 12

REMINDER

MIDTERM
Monday, October 25

Limit of a Function

- Suppose $f(x, y)$ is a function with domain D and $(x_0, y_0) \in D$
- $$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L,$$
  means the following: If $(x_1, y_1), (x_2, y_x), \ldots$ is any sequence of points such that
  $$\lim_{k\to\infty} (x_k, y_k) = (x_0, y_0),$$
  then
  $$\lim_{k\to\infty} f(x_k, y_k) = L$$

Informal Rules on Limits

- Consider
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$$
- Assume f is defined by a single formula
- First, try substituting (x_0, y_0) into the formula for f
  - If $f(x_0, y_0)$ has a valid value, then the limit is $f(x_0, y_0)$
  - If $f(x_0, y_0)$ is equal to an undefined expression such as
  $$\frac{\text{nonzero}}{0}, \ \sqrt{\text{negative}}, \text{ or } \log(\text{nonpositive}),$$
  then the limit is undefined
  - If $f(x_0, y_0)$ is equal to an indeterminate expression such as
  $$\frac{0}{0} \text{ or } \frac{\infty}{\infty},$$
  then you have to investigate further

Examples of Limits

- $\lim_{(x,y)\to(1,0)} \frac{x^2+y^2}{x^2-y^2} = 1$
- $\lim_{(x,y)\to(1,-1)} \frac{x^2+y^2}{x^2-y^2}$ is undefined
- $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2}$ is indeterminate
  - Need to investigate further

Detection of Undefined Limits

- Consider
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$$
- If there is a sequence $(x_k, y_k) \to (x_0, y_0)$ such that every $f(x_k, y_k)$ is undefined, then the limit is undefined
- If there is a sequence $(x_k, y_k) \to (x_0, y_0)$ such that the sequence $f(x_k, y_k)$ has no limit, then the limit is undefined then the limit is undefined
- If there is a sequence
$$(x_k, y_k) \to (x_0, y_0) \text{ such that } f(x_k, y_k) \to L$$
and another sequence
$$(\tilde{x}_k, \tilde{y}_k) \to (\tilde{x}_0, \tilde{y}_0) \text{ such that } f(\tilde{x}_k, \tilde{y}_k) \to \widetilde{L},$$
and $L \neq \widetilde{L}$, the the limit is undefined

Examples of Undefined Limits

- Consider $\lim_{(x,y)\to(0,0)} \frac{x+2y}{x^2-y^2}$
  - If $(x_k, y_k) = \frac{1}{k}$, then $(x_k, y_k) \to (0,0)$, but
    $$\frac{x_k + 2y_k}{x_k^2 - y_k^2} = \frac{\frac{1}{k} + \frac{2}{k}}{\frac{1}{k^2} - \frac{1}{k^2}} \text{ is undefined for every } k$$
  - Therefore, the limit is undefined
- Consider $\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{x^2+y^2}$
  - If $(x_k, y_k) = (\frac{1}{k}, 0)$, then $(x_k, y_k) \to (0,0)$ and
    $$\frac{x_k^2 - x_k y_k}{x_k^2 + y_k^2} = \frac{\frac{1}{k^2} - 0}{\frac{1}{k^2} + 0} = 1 \to 1$$
  - If $(x_k, y_k) = (0, \frac{1}{k})$, then $(x_k, y_k) \to (0,0)$ and
    $$\frac{x_k^2 - x_k y_k}{x_k^2 + y_k^2} = \frac{0-0}{0+1} = 0 \to 0$$
  - No limit because answers are inconsistent

Example of a Convergent Limit

- Consider
$$\lim_{(x,y)\to(0,0)} \frac{2x^2y - 5xy^2}{x^2 + 3y^2}$$
- Key observations:
  - Denominator is never zero when $(x, y) \neq (0, 0)$
  - $x^2 + 3y^2 \geq x^2 + y^2$
  - $|x|, |y| \leq \sqrt{x^2 + y^2}$
- For each $(x, y) \neq 0$, the measurement error is
$$\left|\frac{2x^2y - 5xy^2}{x^2 + 3y^2}\right| \leq \frac{2|x|^2|y| + 5|x||y|^2}{x^2 + y^2}$$
$$\leq \frac{7(x^2 + y^2)^{3/2}}{x^2 + y^2} \leq 7(x^2 + y^2)^{1/2}$$
- Since, $\lim_{(x,y)\to(0,0)} \sqrt{x^2 + y^2} = 0$, the limit is zero

Evaluation of a Limit

- Consider

$$\lim_{(x,y)\to(x_0,y_0)} (\text{single formula in } x \text{ and } y)$$

- First, try plugging $(x, y) = (x_0, y_0)$ into the formula
  - If it works, then the answer is the limit
  - If it is undefined, then the limit is undefined
  - If it is indeterminate, need to investigate further

Limit at $(0, 0)$ of a Rational Function

- Consider

$$\lim_{(x,y)\to(0,0)} \frac{\text{polynomial in } x, y}{\text{polynomial in } x, y}$$

- Plug $(x, y) = (0, 0)$ into the formula
  - If the denominator is nonzero, then the limit exists
  - If the denominator is zero and the numerator is nonzero, then the limit does not exist
- Rules of thumb (Added in revised version of these slides):
  - Let p be the lowest degree of terms in the numerator
  - Let p be the lowest degree of terms in the numerator
  - If $p \leq q$, then the limit is undefined
  - If $p > q$, then the limit is 0
- These conclusions need to be justified

First Attempt

- Consider the limit

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{g(x,y)}, \qquad (1)$$

  where f and g are polynomials
- First, plug the sequences

$$(x_k, y_k) = \left(\frac{1}{k}, 0\right) \text{ and } (x_k, y_k) = \left(0, \frac{1}{k}\right)$$

  into (1) and consider

$$\lim_{k\to\infty} \frac{f(x_k)}{g(x_k)} \qquad (2)$$

- If (2) is undefined for either sequence, the (1) is undefined
- If the the limits of (2) are defined for both sequences but are not equal, then the limit (1) is undefined
- If the the limits of (2) are defined for both sequences and are equal, then proceed to next step

Second Attempt

- Let a and b be unspecified constants, to be chosen later and plug the sequence

$$(x_k, y_k) = \left(\frac{a}{k}, \frac{b}{k}\right)$$

  into (2)
- You will get a formula that contains a and b only
- If it is possible to choose values for a and b such that the formula is undefined, then the limit (1) does not exist
  - This will usually happen if $p < q$
- If it is possible to choose values for a and b and get two different values for the formula, then the limit (1) does not exist
  - This will usually happen if $p = q$
- If the formula is a constant that depends on neither a nor b, then proceed to next step

Third Attempt

- Verify that the denominator $g(x, y)$ is never zero, no matter what x and y are
- Key facts to use:

$$|A + B| \leq |A| + |B|, \;\; |AB| = |A||B|, \;\; |A|, |B| \geq \sqrt{A^2 + B^2}$$

- Use key facts to find a constant $d > 0$ such that the denominator satisfies

$$|g(x, y)| \geq d(x^2 + y^2)^{q/2} \text{ if } (x, y) \neq (0, 0)$$

and a constant $c > 0$ such that the numerator satisfies

$$|g(x, y)| \leq c(x^2 + y^2)^{p/2} \text{ for all } (x, y)$$

- Therefore, since $p < q$,

$$\left| \lim_{(x,y)\to(0,0)} \frac{f(x, y)}{g(x, y)} \right| \leq \lim_{(x,y)\to(0,0)} \frac{c(x^2 + y^2)^{p/2}}{d(x^2 + y^2)^{q/2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{c}{d}(x^2 + y^2)^{(p-q)/2} = 0$$

If none of the above works, move on to another problem

Continuity of a function

- A function f is continuous, if it never jumps suddenly in value
- A function f is continuous at a point (x_0, y_0) in its domain, if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

- A continuous function has a continuous graph with no sudden jumps
- The function $f(x,y) = \sqrt{1 - x^2 + y^2}$ is continuous for all (x,y) in the domain of f
- The function

$$f(x,y) = \begin{cases} \sqrt{1 + x^2 + y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise} \end{cases}$$

is not

Continuous Extension of a Function

- The function

$$f(x,y) = \frac{2x^2y - 5xy^2}{x^2 + 3y^2}$$

  is defined and continuous for all $(x,y) \neq (0,0)$

- However, since

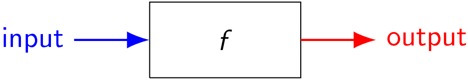
$$\lim_{(x,y)\to(0,)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{2x^2y - 5xy^2}{x^2 + 3y^2} = 0,$$

  the function

$$g(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = 0 \end{cases}$$

  is defined and continuous for all (x,y)

Derivative of a Function



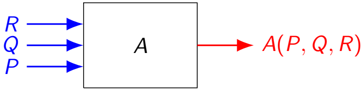
- The derivative of a function measures the senstivity of the function to a small change in input

$$f'(\text{input}) \simeq \frac{\text{change in output}}{\text{change in input}}$$

- A constant function has sensitivity zero
- A linear function has constant sensitivity
- The sensitivity of a nonlinear function depends on the iput
- Given $f(1) = 3$ and $f'(1) = -2$, estimate $f(0.8)$

$$\begin{aligned}
\text{Change in output} &\simeq (\text{sensitivity})(\text{change in input}) \\
&= (-2)(-0.2) = 0.4 \\
f(0.8) - f(1) &\simeq (\text{sensitivity})(\text{change in input}) \\
&= (-2)(0.8 - 1) = 0.4 \\
f(0.8) &\simeq f(1) + (f(0.8) - f(1)) = 3 + 0.4 = 3.4
\end{aligned}$$

Partial Derivatives



- Sensitivity to a small change in one of the inputs, while keeping the others unchanged

$$\frac{\partial A}{\partial Q}(P_0, Q_0, R_0) \simeq \frac{\text{small change in A}}{\text{small change in } Q} = \frac{A(P_0, Q_1, R_0) - A(P_0, Q_0, R_0)}{Q_1 - Q_0}$$

- Total change in A estimated by adding partial changes

$$A(P_1, Q_1, R_1) - A(P_0, Q_0, R_0)$$
$$\simeq \frac{\partial A}{\partial P}(P_0, Q_0, R_0)(P_1 - P_0) + \frac{\partial A}{\partial Q}(P_0, Q_0, R_0)(Q_1 - Q_0)$$
$$+ \frac{\partial A}{\partial R}(P_0, Q_0, R_0)(R_1 - R_0)$$

Calculating Partial Derivatives

- Shorthand: $A_Q = \partial_Q A = \dfrac{\partial A}{\partial Q}$
- To calculate $\partial_Q A$, treat all other variables as constants and differentiate with respect to Q
- If $A(P, Q, R) = P^4 Q - 5Q^3R^2 + e^{-PQR}$, then

$$\partial_P A = 4P^3Q + e^{-PQR}(-QR) = 4P^3Q - QRe^{-PQR}$$
$$\partial_Q A = P^4 - 15Q^2R^2 - PRe^{-PQR}$$
$$\partial_R A = -10Q^3R - PQe^{-PQR}$$

Higher Order Derivatives

If $P(A, B) = e^{AB^2 - A^2 B}$,

$$
\begin{aligned}
P_A &= (B^2 - 2AB)e^{AB^2 - A^2 B} \\
P_B &= (2AB - A^2)e^{AB^2 - A^2 B} \\
P_{AA} &= (-2B)e^{AB^2 - A^2 B} + (B^2 - 2AB)(B^2 - 2AB)e^{AB^2 - A^2 B} \\
&= (-2B + (B^2 - 2AB)^2)e^{AB^2 - A^2 B} \\
P_{AB} &= (-2A)e^{AB^2 - A^2 B} + (B^2 - 2AB)(2AB - A^2)e^{AB^2 - A^2 B} \\
&= (-2A + (B^2 - 2AB)(2AB - A^2))e^{AB^2 - A^2 B} \\
P_{BA} &= (-2A)e^{AB^2 - A^2 B} + (2AB - A^2)(B^2 - 2AB)e^{AB^2 - A^2 B} \\
&= (-2A + (2AB - A^2)(B^2 - 2AB))e^{AB^2 - A^2 B} \\
P_{BB} &= (2AB - A^2)^2 e^{AB^2 - A^2 B}
\end{aligned}
$$

Mixed Partial Commute

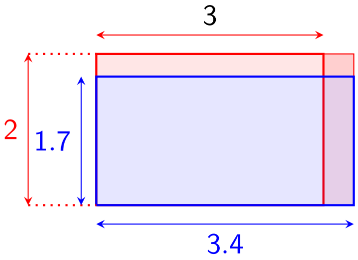
- If a function, its partial derivatives, and its second partial derivatives are all continuous, then it does not matter which order the second partials are calculated
- Given a function $Q(B, C, D)$,

$$\partial_C(\partial_B Q) = \partial_B(\partial_C)B$$
$$(Q_C)_D = (Q_D)_C$$
$$(Q_B)_D = (Q_D)_B$$

- Example: $f(x, y) = xe^{xy}$

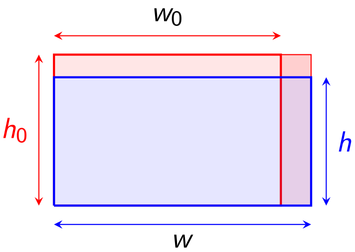
$$f_x = 1(e^{xy}) + x(ye^{xy}) = (1 + xy)e^{xy}$$
$$(f_x)_y = x(e^{xy}) + (1 + xy)(xe^{xy}) = (2x + x^2y)e^{xy}$$
$$f_y = x(xe^{xy}) = x^2e^{xy}$$
$$(f_y)_x = (2x)(e^{xy}) + x^2(ye^{xy}) = (2x + x^2y)e^{xy}$$

Estimate Area of a Box



- Estimate the area of a box with dimensions 3.4 by 1.7;
- Zeroth order estimate: $A = 3(2) = 6$
- First order estimate:
  $A = 6 - 3(0.3) + 2(0.4) = 6 - 0.9 + 0.8 = 5.9$
- Exact answer: $3.4(1.7) = 5.78$

Estimate Area of a Box



- Estimate the area of a box with dimensions w by h
- Zeroth order estimate: $A_0 = w_0 h_0$
- First order estimate: $A_1 = A_0 + w_0(h - h_0) + h_0(w - w_0)$
- Exact answer:

$$
\begin{aligned}
A &= (w_0 + (w - w_0))(h_0 + (h - h_0)) \\
&= w_0 h_0 + w_0(h - h_0) + h_0(w - w_0) + (w - w_0)(h - h_0) \\
&= A_1 + (w - w_0)(h - h_0)
\end{aligned}
$$

Linear Approximation

- Suppose (x, y) is close to (x_0, y_0)
- Given a function $f(x, y)$,

$$
\begin{aligned}
f(x, y) &\simeq f(x_0, y_0) \\
&\qquad + (\text{change in } f \text{ due to change in } x) \\
&\qquad + (\text{change in } f \text{ due to change in } y) \\
&\simeq f(x_0, y_0) + (f_x(x_0, y_0))(x - x_0) + (f_y(x_0, y_0))(y - y_0)
\end{aligned}
$$

- Right side, with (x_0, y_0) held fixed, is a linear function of (x, y)

$$L(x, y) = f(x_0, y_0) + (f_x(x_0, y_0))(x - x_0) + (f_y(x_0, y_0))(y - y_0)$$

- This is called the *linear approximation of f at* (x_0, y_0)
- Note similarity to

$$z = c + a(x - x_0) + b(y - y_0)$$

Example of Linear Approximation

- Suppose $f(x, y) = \sqrt{x^2 + y^2}$
- $f(3, 4) = 5$
- $f(3.1, 3.9) = ?$
- $f_x = \partial_x((x^2 + y^2)^{1/2}) = \dfrac{x}{\sqrt{x^2 + y^2}}$
  - $f_x(3, 4) = \dfrac{3}{5}$
- $f_y = \partial_y((x^2 + y^2)^{1/2}) = \dfrac{y}{\sqrt{x^2 + y^2}}$
  - $f_y(3, 4) = \dfrac{4}{5}$
- Therefore,

$$\begin{aligned} f(3.1, 3.9) &\simeq f(3, 4) + f_x(3, 4)(3.1 - 3) + f_y(3, 4)(3.9 - 4) \\ &= 5 + \frac{3}{5}(0.1) + \frac{4}{5}(-0.1) \\ &= 5 + 0.6 - 0.8 = 4.8 \end{aligned}$$

Example of Linear Approximation

- Suppose

$$f(1,2) = 2,\ \ f_x(1,2) = 3,\ \ f_y(1,2) = 5$$

- Estimate $f(0.8, 2.1)$
- The zeroth order approximation is

$$f(0.8, 2.1) \simeq f(1,2) = 2$$

- The first order (or linear) approximation is

$$\begin{aligned} f(0.8, 2.1) &\simeq f(1,2) + f_x(1,2)(0.8-1) + f_y(1,2)(2.1-2) \\ &= 2 + 3(-0.2) + 5(0.1) \\ &= 2 - 0.6 + 0.5 \\ &= 1.9 \end{aligned}$$

Another Example of Linear Approximation

- Suppose we want to estimate $Q(-0.1, 1.12)$, where

$$Q(A, B) = Be^{2AB}$$

- $(-0.1, 1.12)$ is close to $(0, 1)$ and $Q(0, 1) = 1$
- Partials of Q are $Q_A = 2B^2e^{AB}$ and $Q_B = e^A + 2BAe^{AB}$
- Therefore,

$$\begin{aligned}
Q(0,1) &= 1 \\
Q_A(0,1) &= 2 \\
Q_B(0,1) &= 1 \\
Q(-0.1, 1.12) &\simeq Q(0,1) + Q_A(0,1)(-0.1 - 0) \\
&\quad + Q_B(0,1)(1.12 - 1) \\
&= 1 - 2(0.1) + 0.12 \\
&= 0.92
\end{aligned}$$