# MATH-UA 123 Calculus 3: Contours, Limits, Continuity, Partial Derivatives

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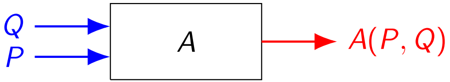
September 29, 2021

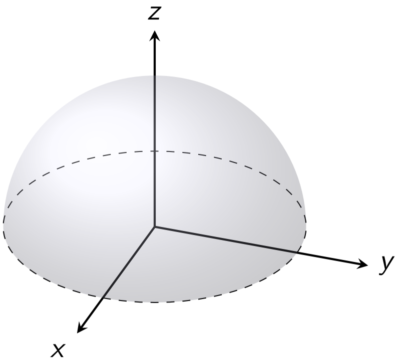
# START RECORDING

# Functions of One or More Variables



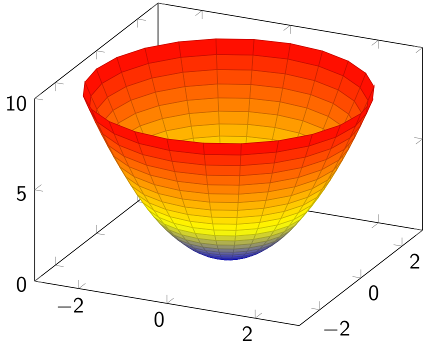

- Input: One or more numbers
- Output: One or more numbers
- $R = A(P, Q)$
  - $A$ is a function that takes two inputs and produces a single output
  - In this formula, the inputs have been named $P$ and $Q$
  - $A(P, Q)$ is the output produced if the inputs are named $P$ and $Q$
  - $R$ is a variable that is set to the output of $A$
- Example: Define $A(S, T) = S^2 + ST - T^2$
- $A(2, 3) = 4 + 6 - 9 = 1$
- $A(P, Q) = P^2 + PQ - Q^2$
- $A(T, S) = T^2 + TS - S^2$
- $A(S + T, S) = (S + T)^2 + (S + T)S - S^2$

# Graph of a Function with 2 Inputs and 1 Output



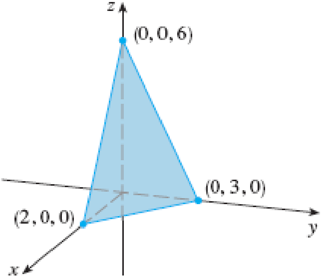

- Graph of $f(x, y) = \sqrt{1 - x^2 - y^2}$
  - $z = \sqrt{1 - x^2 - y^2}$
  - $x^2 + y^2 + z^2 = 1$ and $z \geq 0$
- Domain: $\{x^2 + y^2 \leq 1\}$
- Range: $\{0 \leq z \leq 1\}$

# Elliptic Paraboloid



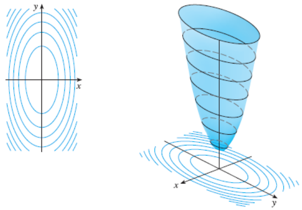

- Graph of $h(x, y) = x^2 + y^2$
- Surface: $z = x^2 + y^2$

# Plane




- Graph of $f(x, y) = 6 - 3x - 2y$
- Domain: $\{0 \leq x \leq 2,\ 0 \leq y \leq -\frac{3}{2}x + 3\}$
- Surface: $3x + 2y + z = 6$

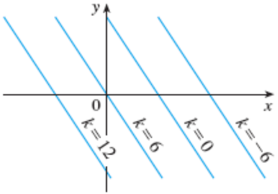
# Contours or Level Sets of a Function



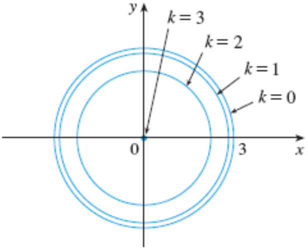

(a) Contour map (b) Horizontal traces are raised level curves

- Contour of a function $f$ at height $h$ is intersection of graph with the plane $z = h$
  - { (x,y) : f(x,y) = y }
- Example: $f(x, y) = 4x^2 + y^2 + 1$
  - $f = 0$: $4x^2 + y^2 + 1 = 0$, which is empty
  - $f = 1$: $4x^2 + y^2 + 1 = 1$, which is a single point at $(0, 0, 1)$
  - $f = 2$: $4x^2 + y^2 + 1 = 2$, which is an ellipse
  - $f = 3$: $4x^2 + y^2 + 1 = 3$, which is a larger ellipse
  - $f = h$, where $h > 1$, is an ellipse, which gets larger as $h$ increases

# Contours of Linear and Quadratic Functions

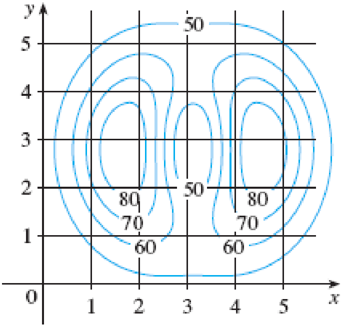



$f(x, y) = 6 - 3x - 2y$



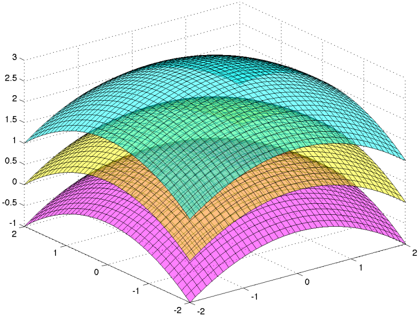

$g(x, y) = 9 - x^2 - y^2$

# Estimating the Value of a Function




- Contours of $z = f(x, y)$
- Shape: Two peaks with valley in between
- Estimate the following:
  - $f(5, 2)$
  - $f(2, 1)$

# Contours of a Function of 3 Variables

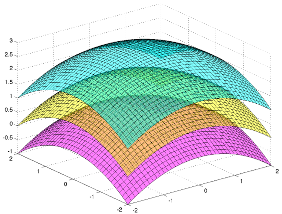



- Cannot draw the graph of a function with 3 inputs
- A contour of a function $f(x, y, z)$ is the surface

$$f(x, y, z) = h, \text{ where } h \text{ is a constant}$$

- Contours can still be used to identify features of the function
- Contours can still be used to estimate the value of the function

# Contours of a Quadratic Function of 3 Variables




$$f(x, y, z) = x^2 + y^2 + 4z$$

- Each contour is given by

  $$x^2 + y^2 + 4z = h,$$

  which is equivalent to

  $$z = \frac{h}{4} - \frac{1}{4}(x^2 - y^2)$$

  which is an upside down circular paraboloid
- For different values of $h$, same surface shifted vertically

# Margin of Error and Limits

- If $x_0$ is the exact value of a number of point and $x_1$ is the measured value, then the error in the measurement is $|x_1 - x_0|$.
- If $m \geq 0$ is the margin of error, then a measurement $x_1$ is within the margin of error if $|x_1 - x_0| \leq m$
- If $x_1, x_2, \ldots$ is an infinite sequence of numbers or points, then

$$\lim_{k \to \infty} x_k = L$$

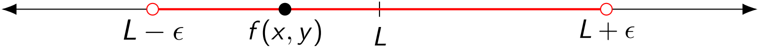
  means:
  - Each $x_k$ is a measurement of the exact value $L$.
  - Given any margin of error $m$, no matter how small, there is a point in the sequence, where *every* measurement after that is within the margin of error
- Formal definition: Given any margin of error $m > 0$, there is an $N > 0$ such that for *every* $k \geq N$,

$$|x_k - L| < m$$

# Limit of a function

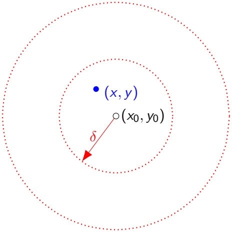
$\lim_{(x,y)\to(0,0)} f(x,y) = L$ means:

If $\epsilon > 0$ is chosen as a margin of error for output of $f$,




then, no matter how small $\epsilon$ is, there is a margin of error for the input to $f$, $\delta > 0$, such that if

$$|(x,y)-(x_0,y_0)| < \delta \text{ but } (x,y) \neq (x_0,y_0),$$




then

$$|f(x,y)-L| < \epsilon$$

# Limit of a Function

- Suppose $f(x, y)$ is a function with domain $D$ and $(x_0, y_0) \in D$
- $$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L,$$
  means the following: If
  $$\lim_{k\to\infty} (x_k, y_k) = (x_0, y_0),$$
  then
  $$\lim_{k\to\infty} f(x_k, y_k) = L$$

# Continuity of a function

- A function f is continuous, if it never jumps suddenly in value
- A function $f$ is continuous at a point $(x_0, y_0)$ in its domain, if

$$\lim_{(x,y)\to(x_\infty,y_\infty)} f(x,y) = f(x_0,y_0)$$

- A continuous function has a continuous graph with no sudden jumps
- The function $f(x,y) = \sqrt{1 - x^2 + y^2}$ is continuous for all $(x,y)$ in the domain of $f$
- The function

$$f(x,y) = \begin{cases} \sqrt{1 + x^2 + y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise} \end{cases}$$

is not

# Examples of Limits

- $\lim_{(x,y)\to(0,0)} \frac{x^2-xy+1}{y^2-3} = -\frac{1}{3}$
- $\lim_{(x,y)\to(0,0)} \frac{x+2y}{x^2-y^2}$
  - No limit, because if $y_k = x_k$, then the formula is undefined
- $\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{x^2+y^2}$
  - If $x_k \to 0$, where every $x_k \neq 0$, and $y_k = 0$, then

$$\lim_{k\to\infty} \frac{x_k^2 - x_k y_k}{x_k^2 + y_k^2} = \lim_{k\to\infty} \frac{x_k^2}{x_k^2} = \lim_{k\to\infty} 1 = 1$$

  - If $x_k = 0$ and $y_k \to 0$, where every $y_k \neq 0$, then

$$\lim_{k\to\infty} \frac{x_k^2 - x_k y_k}{x_k^2 + y_k^2} = \lim_{k\to\infty} \frac{0}{y_k^2} = \lim_{k\to\infty} 0 = 0$$

  - No limit because inconsistent answers

# Example of a Convergent Limit

- Claim:

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y - 5xy^2}{x^2 + 3y^2} = 0$$

- Key observations:

$$x^2 + 3y^2 \geq x^2 + y^2 \text{ and } |x|, |y| \leq \sqrt{x^2 + y^2}$$

- For each $(x, y) \neq 0$, the measurement error is

$$\left| \frac{2x^2y - 5xy^2}{x^2 + 3y^2} - 0 \right| \leq \frac{2|x|^2|y| + 5|x||y|^2}{x^2 + y^2}$$

$$\leq \frac{7(x^2 + y^2)^{3/2}}{x^2 + y^2} \leq 7(x^2 + y^2)^{1/2}$$

- Since, $\lim_{(x,y)} \sqrt{x^2 + y^2} = 0$, the limit above is indeed 0

# Continuous Extension of a Function

- The function
$$f(x,y)=\frac{2x^2y-5xy^2}{x^2+3y^2}$$
is defined and continuous for all $(x,y)\neq(0,0)$
- However, since
$$\lim_{(x,y)\to(0,)} f(x,y)=\lim_{(x,y)\to(0,0)}\frac{2x^2y-5xy^2}{x^2+3y^2}=0,$$
the function
$$g(x,y)=\begin{cases} f(x,y) & \text{if } (x,y)\neq 0 \\ 0 & \text{if } (x,y)=0 \end{cases}$$
is defined and continuous for all $(x,y)$

# Evaluation of a Limit

- Consider $\lim_{(x,y)\to(x_0,y_0)}$(formula in $x$ and $y$)
- First, try plugging $(x, y) = (x_0, y_0)$ into the formula
  - If it all works, then the answer is the limit
  - If it is undefined but not an indeterminate form ($0/0$, $\infty/\infty$, or $(0)(\infty)$), then there is no limit
  - If it is an indeterminate form, need to do more work

# Limit at $(0,0)$ of a Rational Function

- Consider $\lim_{(x,y)\to(0,0)} \frac{\text{polynomial in } x,y}{\text{polynomial in } x,y}$
- Plug $(x,y) = (0,0)$ into the formula
  - If the denominator is nonzero, then the limit exists
  - If the denominator is zero and the numerator is nonzero, then the limit does not exist
- Plug in $(x,y) = (x, ax)$ and take the limit $x \to 0$
  - If this limit does not exist or the answer depends on $a$, then there is no limit
  - If the answer is the same, no matter what $a$ is, go to the next step
  - You can also try plugging in $(x,y) = (ay, y)$ and doing the same

# Limit at $(0, 0)$ of a Rational Function, Continued

- See if you can find a positive power $p$ and a positive constant $c$ such that
$$\text{denominator} > c(x^2 + y^2)^p$$
for all $(x, y)$ close to $(0, 0)$.
  - If so, then use that, use the inequalities
$$|x|, |y| \leq (x^2 + y^2)^{1/2}$$
to simplify the numerator, and see if you get a consistent limit (usually, 0)
  - Otherwise, the limit is still unknown and you should move on