MATH-UA 123 Calculus 3: Parameterized Curves, Velocity, Speed, Acceleration, Arclength

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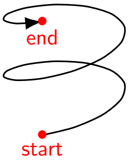
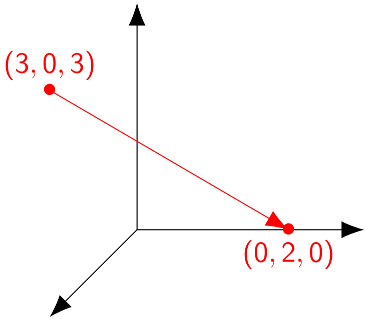
LIVE TRANSCRIPT

Parameterized Curve



- A parameterized curve is a function, whose input is a real number and output is a point in 2-space or 3-space
  - The domain is an interval, which can have finite or infinite length
  - The range is the set of points in space
  - The input is called the *parameter*
- The input is typically called t, and the output is a point in space, typically written as $P(t) = (x(t), y(t), z(t))$.
- Each coordinate is a function of the parameter
- A parameterized curve has an orientation (direction)
- If the parameterized curve has finite size, then there is a start point and an end point

Example: Parameterized line in 3-space

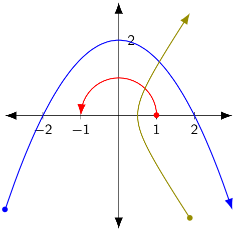


$$\begin{aligned}(x(t), y(t), z(t)) &= (3, 0, 3) + t((0, 2, 0) - (3, 0, 3)) \\ &= (3, 0, 3) + t\langle -3, 2, -3 \rangle \\ &= (3 - 3t, 2t, 3 - 3t)\end{aligned}$$

where the domain is the interval

$$I = [0, 1] = \{t \ : \ 0 \leq t \leq 1\}$$

Examples: Parameterized Curves in 2-space



Parabola: $P(s) = (s, 4 - s^2), \ -3 \leq s \leq 3$

Lies along: $y = 4 - x^2$

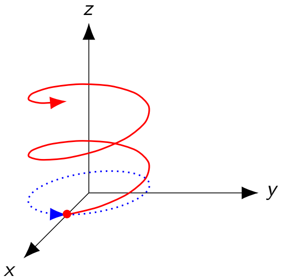
Half circle: $Q(\theta) = (\cos(\theta), \sin(\theta)), \ 0 \leq \theta \leq \pi$

Lies along: $x^2 + y^2 = 1$

Hyperbola: $R(t) = (2\cosh(t), 3\sinh(t)), \ -2 \leq t \leq 2$

Lies along: $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

Helix in 3-space



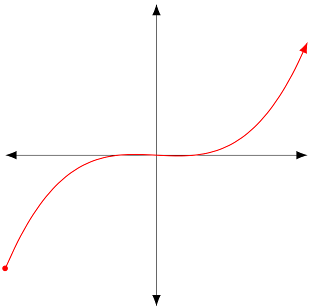
$$(x(s), y(s), z(s)) = (\cos(s), \sin(s), s), \;\; -\infty < t < \infty$$

or

$$(x(c), y(c), z(c)) = \langle \cos(c), \sin(c), c \rangle, \;\; -\infty < c < \infty$$

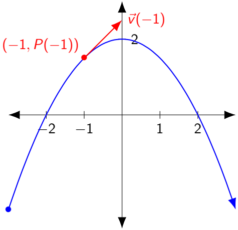
Parameterized Graphs

If you can solve for one of the variables x, y, or z, then you can use that variable as the parameter.



- Equation of curve: $y = (x-1)x(x+1)$, $-4 \leq x \leq 4$
- Parameterization: $(x(t), y(t)) = (t, (t-1)t(t+1))$

Velocity



- Velocity at $P(t)$ is the rate of change of the parameterized curve

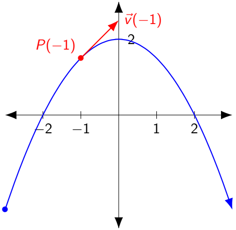
$$\vec{v}(t) \simeq \frac{\text{Change in position}}{\text{Change in time}} \simeq P'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

- Also written as

$$\vec{v}(t) = \dot{P}(t) = \langle \dot{x}(t), \dot{y}(t), \dot{z}(t) \rangle$$

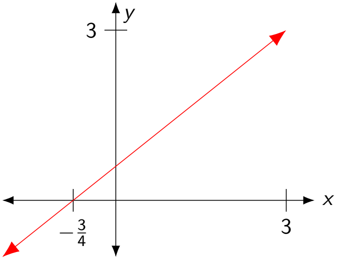
- The velocity of a parameterized curve is a vector-valued function
  - Domain is same as for the parameterized curve
  - Range is the set of vectors

Velocity of a Parameterized Parabola



- $y = 4 - x^2$
- $(x(s), y(s), z(s)) = (s, 4 - s^2),\ -3 \leq s \leq 3$
- Velocity: $\vec{v}(t) = \langle \dot{x}(s), \dot{y}(s) \rangle = \langle 1, -2s \rangle,\ -3 \leq s \leq 3$
- The x coordinate increases at same rate as the parameter s
- The y coordinate increases when $-3 \leq s \leq 0$ and decreases when $0 \leq s \leq 3$

Velocity of a Parameterized Line



- Consider the line $4x - 5y + 3 = 0$
- Two points on the line are $(3, 3)$ and $(8, 7)$
- If the parameterization is

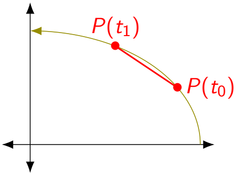
  $$(x(t), y(t)) = (3 + 5t, 3 + 4t),$$

  the velocity is $\vec{v}(t) = \langle 5, 4 \rangle$
- If the parameterization is

  $$(x(t), y(t)) = (8 - 15t + 10t^2, 7 - 12t + 8t^2),$$

  the velocity is $\vec{v}(t) = \langle -15 + 20t, -12 + 16t \rangle$

Speed



- Speed = rate of change of distance along parameterized curve
- If t_1 is close to t_0, then the distance along curve between $P(t_0)$ and $P(t)$ is approximately

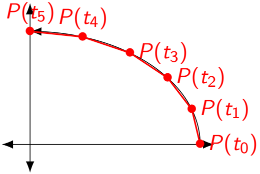
$$d_{\text{curve}}(P(t_0), P(t_1)) \simeq |P(t_1) - P(t_0)|$$

- Therefore, the speed at t_0 is

$$\sigma(t_0) \simeq \frac{d_{\text{curve}}(P(t_0), P(t_1))}{t_1 - t_0} \simeq \left| \frac{P(t_1) - P(t_0)}{t_1 - t_0} \right| \simeq |\vec{v}(t_0)|.$$

- Consider the curve $P(t) = (t, t^2, t^3)$, $-1 \leq t \leq 1$
  - Its velocity is $v(t) = P'(t) = \langle 1, 2t, 3t^2 \rangle$
  - Its speed is $\sigma(t) = |v(t)| = \sqrt{1 + 4t^2 + 9t^4}$

Length of a Curve



- If you drive at constant speed, then the distance traveled is

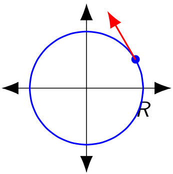
$$\text{distance} = (\text{speed})(\text{time elapsed})$$

- If the curve is divided into small pieces, then its length is

$$\ell = \sum_{k=1}^{N} \ell_k \simeq \sum_{k=1}^{N} \sigma(t_k)(t_k - t_{k-1}) \to \int_{t=t_0}^{t=t_N} \sigma(t)\,dt$$

- **The length of a parameterized curve is equal to the integral of the speed with respect to the parameter**

Length of a Circle



- The angular parameterization of a circle with radius R and centered at the origin is given by

$$P(\theta) = (R\cos(\theta), R\sin(\theta)), \;\; 0 \le \theta \le 2\pi$$

- The velocity and speed are

$$\vec{v}(\theta) = \langle -R\sin(\theta), R\cos(\theta)\rangle$$
$$\sigma(\theta) = |\vec{v}(\theta)| = R$$

Therefore, the length of the circle is

$$\ell = \int_{\theta=0}^{\theta=2\pi} \sigma(\theta)\, d\theta = \int_{\theta=0}^{\theta=2\pi} R\, d\theta = 2\pi R$$

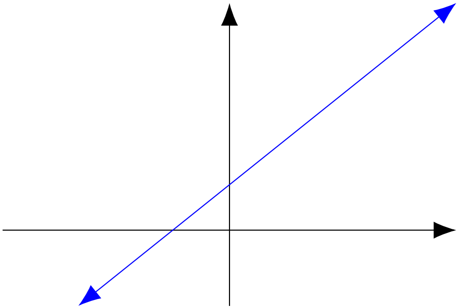
Acceleration

- Acceleration is the rate of change of velocity

$$\vec{a}(t) = \vec{v}\,'(t) = \langle x''(t), y''(t), z''(t) \rangle$$

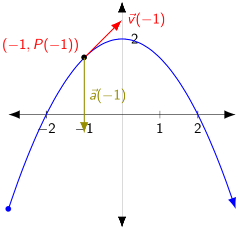
- Acceleration is a vector-valued function with the same domain and range as velocity
  - Input = curve parameter
  - Output is a vector

Acceleration of a Parametrized Line



- $(x(t), y(t)) = (3 + 5t, 3 + 4t)$
  - $\vec{v}(t) = \langle 5, 4 \rangle$
  - $\vec{a}(t) = \vec{v}\,'(t) = 0$
- $(x(t), y(t)) = (8 - 15t + 10t^2, 7 - 12t + 8t^2)$
  - $\vec{v}(t) = \langle -15 + 20t, -12 + 16t \rangle$
  - $\vec{a}(t) = \vec{v}\,'(t) = \langle 20, 16 \rangle$
  - Acceleration is parallel to velocity

Acceleration of Parameterized Parabola

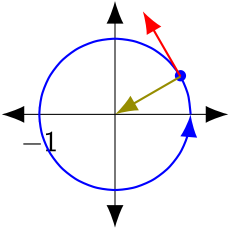


- For any $-3 \leq s \leq 3$,

$$P(s) = (x(s), y(s)) = (s, 4 - s^2)$$
$$\vec{v}(s) = \langle \dot{x}(s), \dot{y}(s) \rangle = \langle 1, -2s \rangle$$
$$\vec{a}(s) = (\ddot{x}(s), \ddot{y}(s)) = \langle 0, -2 \rangle$$

- The acceleration does *not* point in the same direction as velocity

Acceleration of Angular Parameterization of Circle



- $P(\theta) = (R\cos(\theta), R\sin(\theta))$
- $\vec{v}(\theta) = \langle -R\sin(\theta), R\cos(\theta)\rangle$
- $s(\theta) = R$
- $\vec{a}(\theta) = -R\langle\cos\theta, \sin\theta\rangle$
- The speed is constant, so acceleration in the direction of the curve is zero
- The acceleration is orthogonal to velocity
- The change in velocity is due completely to change in direction of the curve

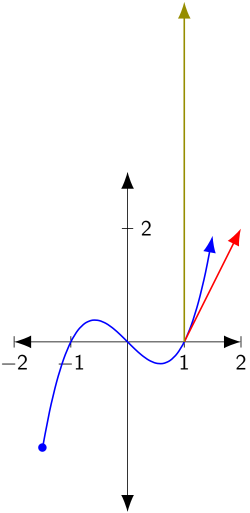
Tangential and Normal Acceleration

- Consider a parameterized curve $P(t)$ with velocity $\vec{v}(t)$, speed $\sigma(t) = |\vec{v}(t)|$, and acceleration $\vec{a}(t) = \vec{v}\,'(t)$
- Let $\vec{u}(t) = \dfrac{\vec{v}(t)}{|\vec{v}(t)|}$ be the unit vector in the direction of velocity
- $\vec{v}(t) = |\vec{v}(t)| \dfrac{\vec{v}(t)}{|\vec{v}(t)|} = \sigma(t)\vec{u}(t)$
- $\vec{a}(t) = \vec{v}\,'(t) = \sigma'(t)\vec{u}(t) + \sigma(t)\vec{u}\,'(t)$
- Since $|\vec{u}(t)|^2 = 1$ for any t,

$$0 = \frac{d}{dt}|\vec{u}(t)|^2 = \frac{d}{dt}(\vec{u} \cdot \vec{u}) = 2\vec{u} \cdot \vec{u}\,'$$

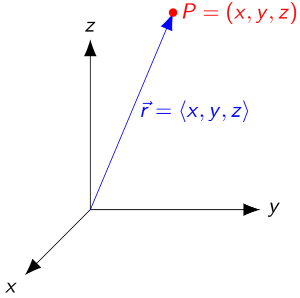
- Therefore, the acceleration is the sum of two terms, one parallel to velocity and one orthogonal to velocity
- $c_{\vec{v}}(\vec{a}) = \sigma'(t)$
- $\vec{a} - p_{\vec{v}}(\vec{a}) = \sigma(t)\vec{u}'$

Velocity, Speed, and Acceleration of a Parameterized Graph



- $y = f(x)$
- $P(t) = (t, f(t))$
- $\vec{v}(t) = \langle 1, f'(t) \rangle$
- $s(t) = \sqrt{1 + (f'(t))^2}$
- $\vec{a}(t) = \langle 0, f''(t) \rangle$

Position Vector



- The position vector of a point is the vector from the origin to the point
- $\vec{r} = P - O$ and $P = O + \vec{r}$
- If $P = (x, y, z)$, then

$$\vec{r} = \langle x, y, z \rangle = \vec{i}x + \vec{j}y + \vec{k}z$$

- From now on, we will often use the position vector $\vec{r}$ instead of points P

Reconstruct Parameterized Curve from Velocity

- Suppose $\vec{r}(t)$ is a parameterized curve, $\vec{v}(t) = \vec{r}\,'(t)$ its velocity
- If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, then

$$x'(t) = \vec{v_1}(t),\ y'(t) = \vec{v_2}(t),\ z'(t) = \vec{v_3}(t)$$

- By the Fundamental Theorem of Calculus, the curve $\vec{r}(t)$, can be reconstructed from using one point on the curve and the velocity function $\vec{v}(t)$ for all t,

$$x(t) - x(t_0) = \int_{s=t_0}^{s=t} v_1(s)\, ds$$

$$y(t) - y(t_0) = \int_{s=t_0}^{s=t} v_2(s)\, ds$$

$$z(t) - z(t_0) = \int_{s=t_0}^{s=t} v_3(s)\, ds$$

- In short, $\vec{r}(t) = \vec{r}(t_0) + \displaystyle\int_{s=t_0}^{s=t} \vec{v}(s)\, ds$

Reconstruct Velocity from Acceleration

- If the velocity is $\vec{v}(t) = \langle v_1(t), v_2(t), v_3(t) \rangle$, then the acceleration is $\vec{a}(t) = \vec{v}\,'(t)$
- By the Fundamental Theorem of Calculus, we can recover the velocity function $\vec{v}(t)$, if we know both the initial velocity $\vec{v}(t_0)$ and the acceleration function $\vec{a}(t)$,

$$v_1(t) - v_1(t_0) = \int_{s=t_0}^{s=t} a_1(s)\,ds$$

$$v_2(t) - v_2(t_0) = \int_{s=t_0}^{s=t} a_2(s)\,ds$$

$$v_3(t) - v_3(t_0) = \int_{s=t_0}^{s=t} a_3(s)\,ds$$

- This can be written more tersely as

$$\vec{v}(t) = \vec{v}(t_0) + \int_{s=t_0}^{s=t} \vec{a}(s)\,ds$$

Reconstruct Curve from Acceleration

- A curve can be reconstructed, if we know the following
  - Where the curve starts, $P(t_0) = (x(t_0), y(t_0), z(t_0))$
  - The initial velocity, $\vec{v}(t_0)$
  - The acceleration function $\vec{a}(t)$, for all t
- Example:

$$
\begin{aligned}
P(0) &= (1, -1, 2) \\
\vec{v}(0) &= \langle 0, 1, 0 \rangle \\
\vec{a}(t) &= \langle 2 - 6t, 2 + 6t, 2 \rangle \\
\implies \quad \vec{v}(t) &= \langle 0, 1, 0 \rangle + \int_{\tau=0}^{\tau=t} \langle 2 - 6\tau, 2 + 6\tau, 2 \rangle \, d\tau \\
&= \langle 2t - 3t^3, 1 + 2t + 3t^2, 2t \rangle \\
P(t) &= (1, -1, 2) + \int_{\tau=0}^{\tau=t} \langle 2\tau - 3\tau^2, 1 + 2\tau + 3\tau^2, 2\tau \rangle \, d\tau \\
&= (1, -1, 2) + \langle t^2 - t^3, t + t^2 + t^3, t^2 \rangle \\
&= (1 + t^2 - t^3, -1 + t + t^2 + t^3, 2 + t^2)
\end{aligned}
$$

Motion of an Object in Space

- No prior knowledge of physics is needed!
- Suppose the parameter t represents time and $\vec{r}(t)$ is the position of an object in space at time t.
- $\vec{r}(t)$ describes the path of the object as a functio of time t
- The velocity and acceleration of the object are $\vec{v}(t) = \vec{r}\,'(t)$ and $\vec{a}(t) = \vec{v}\,'(t)$
- If we know the velocity $\vec{v}$ as a function of time t and a starting position $\vec{r}(t_0)$, then we can reconstruct the path of the object using

$$\vec{r}(t) = \vec{r}(t_0) + \int_{s=t_0}^{s=t} \vec{v}(s)\, dx$$

- If we know the acceleration $\vec{v}$ as a function of time t and a starting velocity $\vec{v}(t_0)$, then we can reconstruct the velocity of the object using

$$\vec{v}(t) = \vec{v}(t_0) + \int_{s=t_0}^{s=t} \vec{a}(s)\, dx$$

Reconstruction of Path from Acceleration

- Suppose we know the acceleration function $\vec{a}(t)$ of an object moving in space
- Suppose we know the position $\vec{r}(t_0)$ and velocity $\vec{v}(t_0)$ of the object at one specific time t_0
- First, reconstruct the velocity function

$$\vec{v}(t) = \vec{v}(t_0) + \int_{s=t_0}^{s=t} \vec{a}(s)\, dx$$

- Then reconstruct the path of the object

$$\vec{r}(t) = \vec{r}(t_0) + \int_{s=t_0}^{s=t} \vec{v}(s)\, dx$$

Newton's First and Second Laws of Motion and Gravity

- In the absence of any forces, there is no acceleration
  - $\vec{a} = 0$ implies constant velocity
  - Constant velocity implies $\vec{r}(t) = \vec{r}(t_0) + (t - t_0)\vec{v}_0$
- Near the earth's surface, the force of gravity is roughly constant and the acceleration is equal to $-g\vec{k} = \langle 0, 0, -g \rangle$, where $g \simeq 9.8\,\mathrm{m/s^2}$.
- If an object near the earth's surface starts at $\vec{r}(t_0)$ with velocity $\vec{v}(t_0)$, then its velocity and position at any time t is

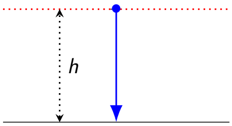
$$\vec{v}(t) = \vec{v}(t_0) + \int_{s=t_0}^{s=t} \vec{k}(-g)\,ds = \vec{v}(t_0) - \vec{k}g(t - t_0)$$

$$\vec{r}(t) = \vec{r}(t_0) + \int_{s=t_0}^{s=t} \vec{v}(s)\,ds$$

$$\vec{r}(t) = \vec{r}(t_0) + \int_{s=t_0}^{s=t} \vec{v}(t_0) - \vec{k}g(s - t_0)\,ds$$

$$= \vec{r}(t_0) + \vec{v}(t_0)(t - t_0) - \vec{k}\frac{1}{2}g(t - t_0)^2$$

Dropped Object



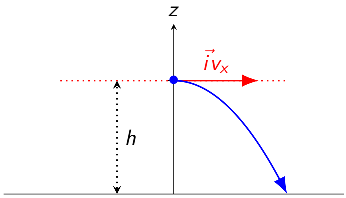
An object is dropped from height h

$$\vec{r}(0) = \vec{k}h$$

$$\vec{v}(0) = \vec{0}$$

$$\begin{aligned} \vec{r}(t) &= \vec{r}(t_0) + \vec{v}(t_0)(t - t_0) - \vec{k}\frac{1}{2}g(t - t_0)^2 \\ &= \vec{k}h - \vec{k}\frac{1}{2}gt^2 \\ &= \vec{k}\left(h - \frac{1}{2}gt^2\right) \end{aligned}$$

Object Shot Horizontally



- An object is shot horizontally with velocity v_x from a height h

$$\vec{r}(0) = h\vec{k}$$

$$\vec{v}(0) = \vec{i}v_x$$

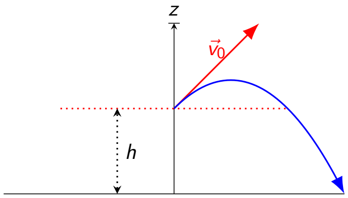
$$\vec{r}(t) = \vec{k}h + \vec{i}v_x t - \vec{k}\frac{1}{2}gt^2$$

Since, for any time t,

$$z = h - \frac{g}{2v_x^2}x^2,$$

the path is an upsidedown parabola starting at the top

Object Shot Upward at an Angle



- An object is shot at a velocity $\vec{v}_0 = \vec{i}v_x + \vec{k}v_z$ from a height h

$$\vec{r}(t) = \vec{k}h + (\vec{i}v_x + \vec{k}v_z)t - \vec{k}\frac{1}{2}gt^2$$
$$= \vec{i}v_x t + \vec{k}\left(h + v_z t - \frac{1}{2}gt^2\right)$$

Since

$$z = h + \left(\frac{v_z}{v_x}\right)x - \left(\frac{g}{2v_x^2}\right)x^2,$$

the path is an upside down parabola