MATH-UA 123 Calculus 3: Quadric Surfaces

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September 22, 2021

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Quadric Surfaces

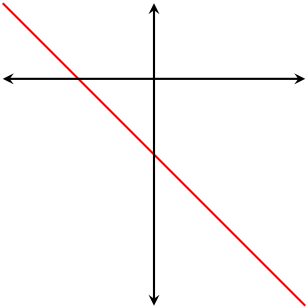
- A quadric surface is a surface in 3-space given by a quadratic polynomial

$$ax^2 + 2bxy + cy^2 + dy^2 + 2eyz + fz^2 + gx + hy + pz = q$$

- First, review quadratic curves in the plane

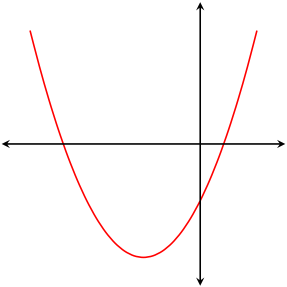
$$ax^2 + by^2 + cx + dy = e$$

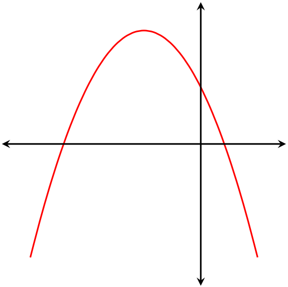
Line



$$ax + by = c$$

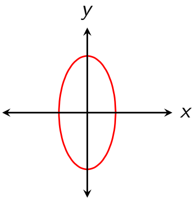
Parabolas





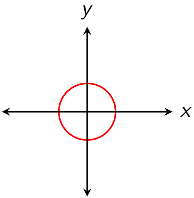
- $y = ax^2 + bx + c$, where $a > 0$
- $y = ax^2 + bx + c$, where $a < 0$

Ellipses and Circles



Ellipse where major and minor axes are the x and y axes:

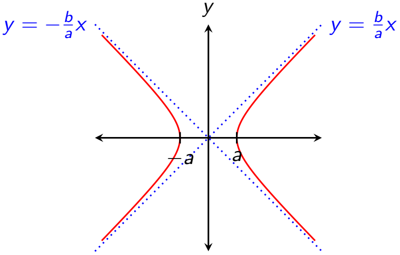
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

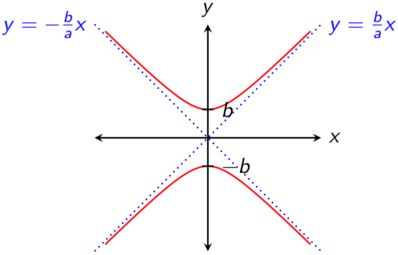


If $a = b$, then a circle of radius a centered at the origin

$$x^2 + y^2 = a^2$$

Hyperbolas

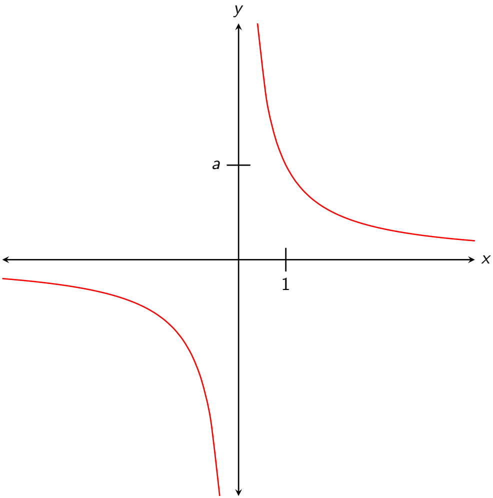




- $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$
- Asymptotes: $y = \pm\dfrac{b}{a}$
- Vertices: $(\pm a, 0)$
- $\dfrac{x^2}{a^2} = 1 + \dfrac{y^2}{b^2}$

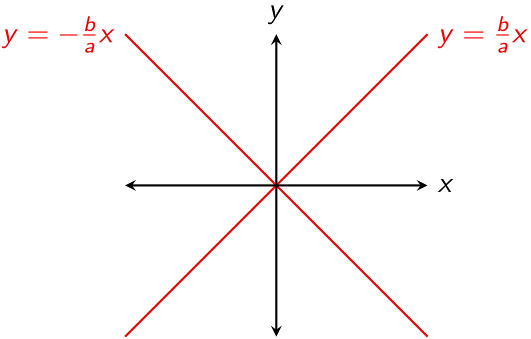
- $-\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
- Asymptotes: $y = \pm\dfrac{b}{a}$
- Vertices: $(0, \pm b)$
- $\dfrac{y^2}{a^2} = 1 + \dfrac{x^2}{b^2}$

Hyperbolas



$$y = \frac{a}{x} \text{ or } xy = a$$

Crossing Lines

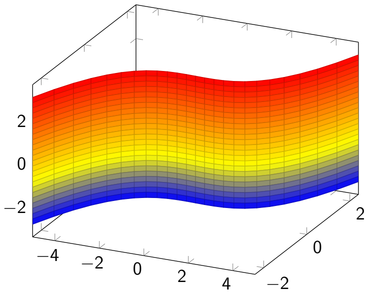


- $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0$
- $\left(\dfrac{x}{a} - \dfrac{y}{b}\right)\left(\dfrac{x}{a} + \dfrac{y}{b}\right) = 0$
- Union of the lines $y = \pm\dfrac{a}{b}x$

Traces of a Surface

- A trace is the intersection of the surface with a plane
- It is usually either a curve or a point
- Drawing the traces for different planes usually gives a good picture of the surface
- Usually use planes parallel to one of the coordinate planes
  - $z = c$ for different values of c
  - $y = c$ for different values of c
  - $x = c$ for different values of c

Cylinders

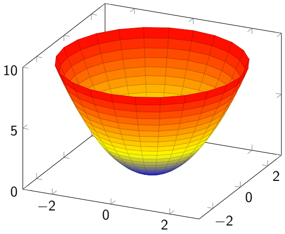


- The surface

$$S = \{(x, y, z) \;:\; f(x, y) = c\}$$

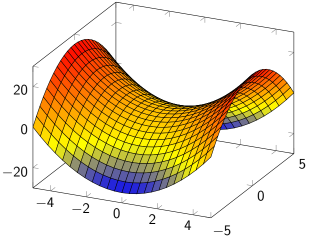
  consists of the union of vertical lines that pass through the curve $f(x, y) = c$ in the xy-plane
- S is an exampe of a ruled surface (a surface that is a union of lines)

Elliptic Paraboloids



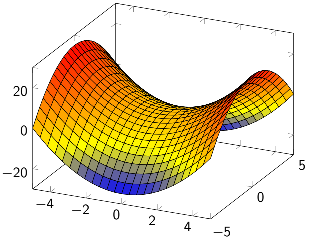
- Graph: $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$
- Trace of $z = c$ is the curve $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = c$
- If $c < 0$, there is no trace, so surface lies above xy-plane
- If $c = 0$, the trace is only the point $(0, 0)$, so surface touches xy-plane only at the origin
- If $c > 0$, the trace is an ellipse, and as c increases, the ellipse increases in size

Hyperbolic Paraboloid



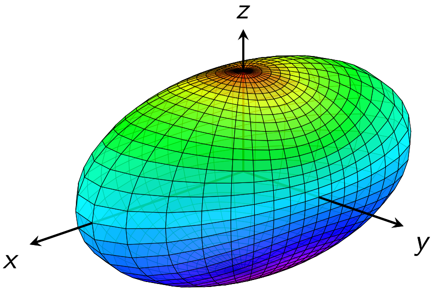
- Graph: $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$
- Trace of $z = c$ is the curve $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = c$
- If $c = 0$, the trace is the intersectio of two lines
- If $c > 0$, the trace is a hyperbola with asymptotes $y = \pm\dfrac{b}{a}x$ and containing the points $(\pm a\sqrt{c}, 0)$
- If $c < 0$, the trace is a hyperbola with asymptotes $y = \pm\dfrac{b}{a}x$ and containing the points $(0, \pm b\sqrt{c})$

Hyperbolic Paraboloid



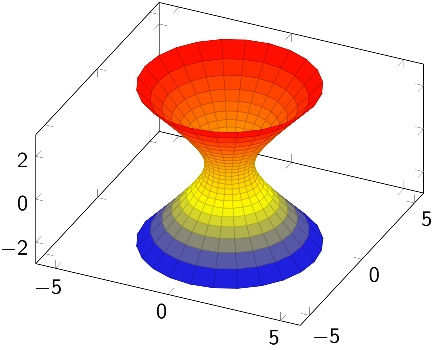
- Trace of $y = c$ is the curve $z = \frac{x^2}{a^2} - \frac{c^2}{b^2}$, which is an upright parabola. When $c = 0$, the bottom of the parabola touches the origin. As c^2 increases, the parabola moves downward
- Trace of $x = c$ is the curve $\frac{c^2}{a^2} - \frac{y^2}{b^2} = z$, which is an upside down parabola. When $c = 0$, the top of the parabola touches the origin. As c^2 increases, the parabola moves upward
- The surface is saddle-shaped

Ellipsoid



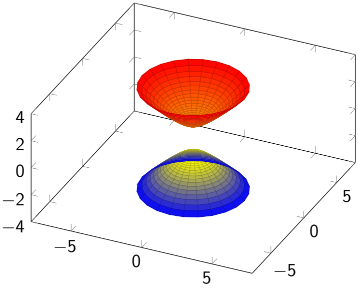
- $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- Contains the points $(\pm a, 0, 0)$, $(0, \pm b, 0)$, $(0, 0, \pm c)$
- If $a = b = c$, then it is a sphere of radius a
- Traces are all ellipses:
  - The trace of $z = h$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{h^2}{c^2}$
  - The trace of $y = h$ is $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 - \frac{h^2}{b^2}$
  - The trace of $x = h$ is $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{h^2}{a^2}$

1-sheeted Hyperboloid



- $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- Trace of $z = h$ is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 + \dfrac{h^2}{c^2}$, which is an ellipse
- Ellipse is smallest when $h = 0$ and increases in size as h grows in magnitude
- Traces of xz-plane and yz-plane are hyperbolas

2-sheeted Hyperboloid



- $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$
- Trace of $z = h$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1 + \dfrac{h^2}{c^2}$
  - If $-c < h < c$, then the trace is empty
  - If $h = c$ or $h = -c$, then the trace is a single point
  - If $h > c$ or $h < -c$, then the trace is an ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{h^2}{c^2} - 1$$

- Traces of xz-plane and yz-plane are hyperbolas