MATH-UA 123 Calculus 3: Parallelopipeds, Lines, Planes

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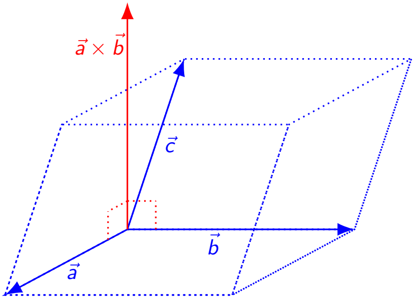
LIVE TRANSCRIPT

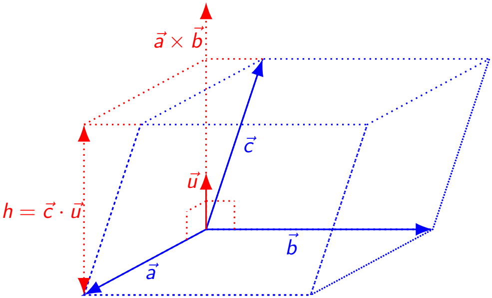
Parallelopiped spanned by 3 Vectors in 3-space



- Three vectors $\vec{a}, \vec{b}, \vec{c}$ are **linearly independent**, if $\vec{c}$ does not lie in the plane containing $\vec{a}$ and $\vec{b}$
- Three linearly independent vectors span a parallelopiped
- An ordered triple of linearly independent vectors, $(\vec{a}, \vec{b}, \vec{c})$, has positive orientation, if it obeys the righthand rule.
- $(\vec{a}, \vec{b}, \vec{c})$ has positive orientation if and only if

$$\vec{c} \cdot (\vec{a} \times \vec{b}) > 0.$$

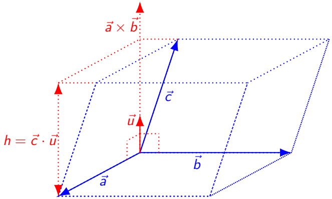
Volume of a Parallelopiped



- Volume = (area of base)(height)
- Area of base $= |\vec{a} \times \vec{b}|$
- Height $= |c_{\vec{u}}(\vec{c})| = |\vec{c} \cdot \vec{u}|$, where

$$\vec{u} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

Unoriented Volume of a Parallelopiped



The unoriented volume V of the parallelopiped is equal to

$$\begin{aligned} |V| &= (\text{area of base})(\text{height}) \\ &= |\vec{a} \times \vec{b}||\vec{c} \cdot \vec{u}| \\ &= |\vec{a} \times \vec{b}| \left| \vec{c} \cdot \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} \right| \\ &= |(\vec{a} \times \vec{b}) \cdot \vec{c}| \end{aligned}$$

Oriented Volume of a Parallelopiped

- Define the oriented volume of a parallelopiped to be

$$V = (\vec{a} \times \vec{b}) \cdot \vec{c}.$$

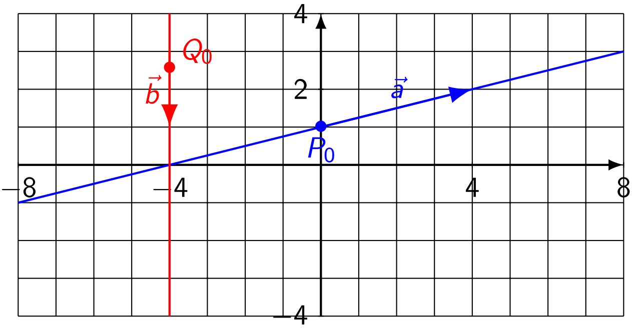
- If $(\vec{a}, \vec{b}, \vec{c})$ has positive orientation, then $V > 0$
- If $(\vec{a}, \vec{b}, \vec{c})$ has negative orientation, then $V < 0$

Lines in 2-space

A line in the xy-plane can be described using equations in at least 3 different ways.

- Graph,
$$y = mx + b,$$
where m is the slope and b is the y-intercept
  - This does not include vertical lines
- Linear equation
$$Ax + By = C$$
  - A vertical line is given by $x = C$
- Linear parameterization: Given any point (x_0, y_0) on the line and nonzero vector $\vec{v} = \langle v_1, v_2 \rangle$ parallel to the line, then, for any scalar t,
$$(x(t), y(t)) = (x_0, y_0) + t\vec{v},$$
lies on the line. Conversely, for any point (x_1, y_1) on the line, there is a scalar t_1 such that
$$(x_1, y_1) = (x_0, y_0) + t\vec{v},$$

Lines in 2-space



$$y = \frac{1}{4}x + 1$$

$$-x + 4y = 4$$

$$(x(t), y(t)) = (0, 1) + t\langle 4, 1\rangle$$

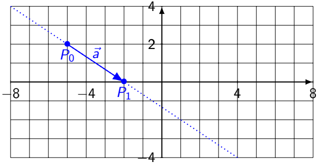
$$P(t) = P_0 + t\vec{a}$$

$$x = -2$$

$$(x(t), y(t)) = (-2, 3) + t\langle 0, -2\rangle$$

$$Q(t) = Q_0 + t\vec{b}$$

Two Distinct Points Uniquely Determine a Line



- Linear equation: The slope is $m = -\frac{2}{3}$ and $P_1 = (-2, 0)$ is a solution to $y = mx + b$. Therefore, the equation is

$$y = -\frac{2}{3}x - \frac{4}{3} \text{ or } 2x + 3y = -4$$

- Linear parameterization: Since

$$P_0 = (-5, 2),\ P_1 = (-2, 0),\ \vec{a} = P_1 - P_0 = \langle 3, -2 \rangle,$$

the linear parameterization

$$P(t) = P_0 + t\vec{a}$$

becomes

$$(x(t), y(t)) = (-5, 2) + t\langle 3, -2 \rangle = (-5 + 3t, 2 - 2t)$$

Two Distinct Points Uniquely Determine a Line

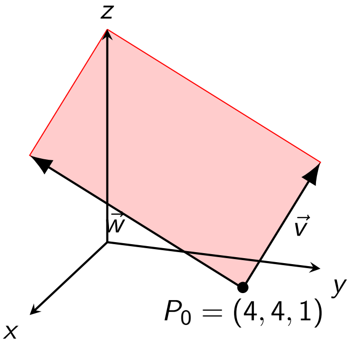
- Given two different points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$, there is a unique line containing them.
- Linear equation:
  - If $x_0 = x_1$, then the line is vertical and given by $x = x_0$
  - Otherwise, you can solve for m and b such that the points P_0 and P_1 are solutions to the equation $y = mx + b$
- A linear parameterization:
  - $P(t) = P_0 + t(P_1 - P_0) = P_0 + t\vec{v}$, where $\vec{v} = P_1 - P_0$
  - $P(0) = P_0$, $P(1) = P_1$

Equations of a Plane in 3-space

A plane in 3-space can be described using equations in at least 3 ways:

- Graph:
  $$z = ax + by + c$$
  - This does not include vertical planes
- Linear equation
  $$Ax + By + Cz = D$$
  - A vertical plane always has an equation of the form
    $$Ax + By = D$$

Linear Parameterization of a Plane in 3-space

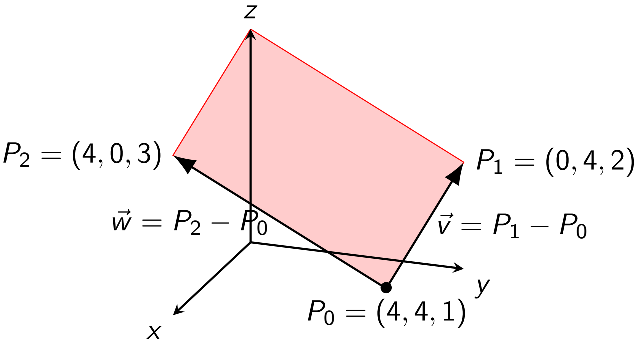


- Let $P_0 = (x_0, y_0, z_0)$ be a point on a plane
- Let $\vec{v}$ and $\vec{w}$ be nonzero non-parallel vectors
- Given any pair of parameters (s, t),

$$P_0 + s\vec{v} + t\vec{w}$$

  lies on the plane
- Conversely, given any point P_1 on the plane, there exists (s_1, t_1) such that

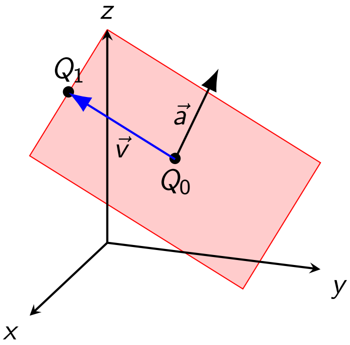
Three Non-collinear Points Uniquely Determine a Plane



- Linear equation: Later
- Linear parameterization:

$$\begin{aligned} P(s,t) &= P_0 + s(P_1 - P_0) + t(P_2 - P_0) \\ &= P_0 + s\vec{v} + t\vec{w} \\ (x(s,t), y(s,t), z(s,t)) &= (4,4,1) + s(-4\vec{i} + \vec{k}) + t(-4\vec{j} + 2\vec{k}) \\ &= (4 - 4s, 4 - 4t, 1 + s + 2t) \end{aligned}$$

Normal Vector to a Plane



- A vector $\vec{v}$ is said to be parallel to a plane, if any line parallel to $\vec{v}$ either lies entirely in the plane or never intersects the plane
- A vector $\vec{a}$ is normal to a plane, if, for any vector $\vec{v}$ that is parallel to the plane, $\vec{a} \cdot \vec{v} = 0$
- A vector $\vec{a}$ is normal to a plane if, for any two points Q_0 and Q_1 in the plane, $\vec{a} \cdot (Q_1 - Q_0) = 0$

Equation of Plane from Its Normal

- Suppose $\vec{a}$ is a vector normal to and Q_0 is a point in a plane A
- A point Q lies in A if and only if $Q - Q_0$ is a vector normal to $\vec{a}$

$$Q \in A \iff \vec{a} \cdot (Q - Q_0) = 0.$$

- Example: Suppose A is a plane containing the point $(1, -1, 0)$ and the vector $2\vec{i} + \vec{k}$ is normal to A. A point $Q = (x, y, z)$ lies in A if and only if

$$\begin{aligned} 0 &= \vec{a} \cdot (Q - Q_0) \\ &= (2\vec{i} + k) \cdot ((x, y, z) - (1, -1, 0)) \\ &= (2\vec{i} + k) \cdot ((x-1)\vec{i} + (y+1)\vec{j} + z\vec{k}) \\ &= 2(x-1) + z \\ &= 2x + z - 2 \end{aligned}$$

- The plane A is the set of solutions to

$$2x + z = 2$$

Normal Vector From Equation of Plane

- Any plane is given by an equation of the form

$$Ax + By + Cz = D$$

- Therefore, given any two points $P_0 - (x_0, y_0, z_0)$ and $P_1 = (x_1, y_1, z_1)$ in the plane,

$$Ax_1 + By_1 + Cz_1 = D$$
$$Ax_0 + By_0 + Cz_0 = D$$

- Subtracting the two equations above, we get

$$A(x_1 - x_0) + B(y_1 - y_0) + C(z - z_0) = 0$$

- If we set $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$, then this becomes

$$\vec{n} \cdot (P_1 - P_0) = 0$$

- Since this holds for any two points in the plane, it follows that $\vec{n}$ is a normal vector to the plane

Upshot: Normal Vector of Plane = Cofficients of Equation

- If a plane is given by an equation $Px + Qy + Rz = S$, then $\vec{n} = P\vec{i}_Q\vec{j} + R\vec{k}$ is normal to the plane
- If the normal vector of a plane Σ is $\vec{n} = \vec{i}n_1 + \vec{j}n_2 + \vec{k}n_3$, then, for any constant D, the plane given by

$$n_1x + n_2y + n_3z = D$$

  is parallel to Σ. There is a unique value of D such that this plane is equal to Σ.
- Example: A normal vector of the plane given by

$$-x + 2y + 5z = 17$$

  is $\vec{n} = -\vec{i} + 2\vec{j} + 5\vec{k}$ and a unit normal vector of

$$\vec{u} = \frac{\vec{n}}{|\vec{n}|} = \frac{-\vec{i} + 2\vec{j} + 5\vec{k}}{\sqrt{30}}.$$

Equation of Plane Given Three Non-Collinear Points

- Suppose P_0, P_1, P_2 are three non-collinear points in 3-space
- A linear parameterization of the plane containing all three points is

$$P(t) = P_0 + s(P_1 - P_0) + t(P_2 - P_0) = P_0 + s\vec{v} + t\vec{w}$$

- Then the vectors $\vec{v} = P_1 - P_0$ and $\vec{w} = P_2 - P_0$ are not parallel
- Therefore, $\vec{v} \times \vec{w} \neq \vec{0}$,
- If $\vec{n} = \vec{v} \times \vec{w} \neq \vec{0}$, then it is normal to the plane. Therefore, an equation of the plane is

$$0 = (\vec{v} \times \vec{w}) \cdot (P - P_0)$$

Example

- Consider three points $A = (1, 1, 1)$, $B = (-2, 0, 1)$, $C = (2, 3, 1)$
- Let Σ be a plane containing A, B, C
- Let $\vec{a} = C - B = \langle 4, 3, 0 \rangle$ and $\vec{b} = A - C = \langle -1, -2, 0 \rangle$.
- A normal to Σ is $\vec{n} = \vec{a} \times \vec{b} = (4\vec{i} + 3\vec{j}) \times (-\vec{i} - 2\vec{j}) = (-8 + 3)\vec{k} = -5\vec{k}$
- Therefore, an equation for Σ is of the form

$$0x + 0y - 5z = D$$

- Since $(1, 1, 1)$ lies in Σ, it follows that $D = -5$ and an equation of the plane is

$$-5z = -5 \text{ or, equivalently, } z = 1$$

- There is only one possible plane D containing A, B, C

Another Example

- Let $P = (4, -8, 16)$, $Q = (4, 0, 8)$, and $R = (4, -6, 14)$
- Let $\vec{v} = R - P = 2\vec{j} - 2\vec{k}$ and $\vec{w} = Q - R = 8\vec{j} - 8\vec{k}$
- A normal to a plane containing P, Q, R is

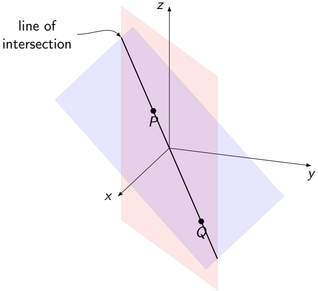
$$\vec{n} = \vec{v} \times \vec{w} = (2\vec{j} - 2\vec{k}) \times (8\vec{j} - 8\vec{k}) = 0$$

- Note that $\vec{v}$ and $\vec{w}$ point in the same direction
- Therefore, P, Q, R lie along a line, and there are infinitely many planes containing them
- Suppose $\vec{n}$ is a nonzero vector normal to $\vec{v}$ and therefore also to $\vec{w}$, such as $\vec{n} = \vec{i}$
- A plane normal to $\vec{n}$ therefore has an equation of the form

$$x = D$$

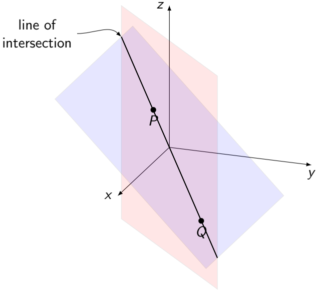
- If $D = 4$, then P, Q, R all lie in the plane
- Any other choice of $\vec{n}$ normal to $\vec{v}$ and $\vec{w}$ also works
- Another possible choice is $\vec{n} = \vec{j} + \vec{k}$

Geometric Description of Line in 3-Space



- Given two distinct points in 3-space, there is a unique line containing them
- The intersection of two non-parallel planes is a line

Description of Line in 3-Space Using Equations



- Linear parameterization: For each $t \in \mathbb{R}$,

$$(x(t), y(t), z(t)) = (x_0, y_0, z_0) + t\langle v_1, v_2, v_3\rangle$$

- The set of all points Q such that

$$\vec{n}_1 \cdot (Q - P) = 0 \text{ and } \vec{n}_2 \cdot (Q - P) = 0$$

- The set of solutions (x, y, z) to

$$a_1x + b_1y + c_1z = d_1 \text{ and } a_2x + b_2y + c_2z = d_2$$

Example: Paramterization

- In general, given two distinct points P and Q, the parameterized line

$$(x, y, z)(t) = P + t(Q - P)$$

satisfies $(x, y, z)(0) = P$ and $(x, y, z)(1) = Q$ and therefore is a line that contains both P and Q

- A parameterized line that contains $(1, 1, 0)$ and $(0, -1, 1)$

$$\begin{aligned}(x, y, z)(t) &= (1, 1, 0) + t((0, -1, 1) - (1, 1, 0)) \\ &= (1, 1, 0) + t\langle -1, -2, 1\rangle \\ &= (1 - t, 1 - 2t, t)\end{aligned}$$

Example: Intersection of Two Planes

- Start with two points, $P = (1, 1, 0)$ and $Q = (0, -1, 1)$
- A plane contains the line through P and Q if its normal is orthogonal to

$$\vec{v} = Q - P = (0, -1, 1) - (1, 1, 0) = \langle -1, -2, 1 \rangle$$

- Two possible normals are

$$\vec{n}_1 = \angle 1, 0, 1 \rangle \text{ and } \vec{n}_2 = \langle 0, 1, 2 \rangle$$

- The first plane has the equation

$$0 = \vec{n}_1 \cdot ((x, y, z) - P) = \langle 1, 0, 1 \rangle \cdot \langle x - 1, y - 1, z \rangle = x - 1 + z$$

- The second plane has the equation

$$0 = \vec{n}_2 \cdot ((x, y, z) - P) = \langle 0, 1, 2 \rangle \cdot \langle x - 1, y - 1, z \rangle = y - 1 + 2z$$

- Therefore, the line is the intersection of the two planes and therefore the set of solutions to

$$x + z = 1 \text{ and } y + 2z = 1$$