MATH-UA 123 Calculus 3: Cross Product

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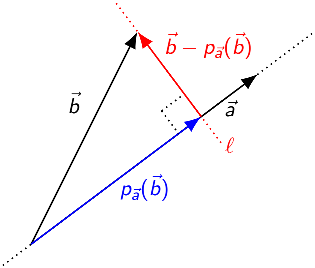
LIVE TRANSCRIPT

Orthogonal Projection



- Given nonzero vectors $\vec{a}$ and $\vec{b}$, the projection of $\vec{b}$ in the direction of $\vec{a}$ is equal to

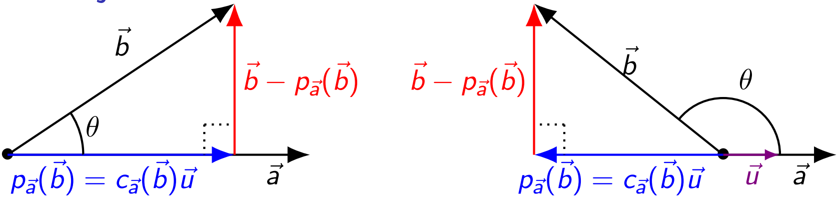
$$p_{\vec{a}}(\vec{b}) = (\vec{b} \cdot \vec{u})\vec{u} = \left((\vec{b} \cdot \left(\frac{\vec{a}}{|\vec{a}|} \right) \right) \frac{\vec{a}}{|\vec{a}|},$$

  where $\vec{u}$ is the unit vector with the same direction $\vec{a}$,

$$\vec{u} = \frac{\vec{a}}{|\vec{a}|}$$

- $\vec{b} - p_{\vec{a}}(\vec{b})$ is orthogonal to $p_{\vec{a}}(\vec{b})$

Scalar Projection



- Since $p_{\vec{a}}(\vec{b})$ is parallel to $\vec{u}$, there is a scalar $c_{\vec{a}}(\vec{b})$ such that

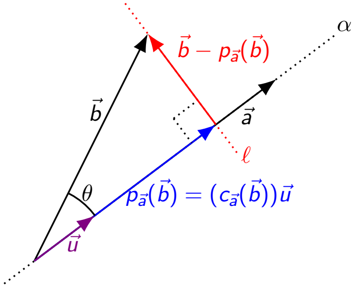
$$p_{\vec{u}}(\vec{b}) = (c_{\vec{u}}(\vec{b}))\vec{u},$$

  where $\vec{u}$ is the direction of $\vec{a}$.
- $c_{\vec{a}}(\vec{b})$ is called the **scalar projection** of $\vec{b}$ onto $\vec{a}$
- If θ is the angle between $\vec{a}$ and $\vec{b}$, then

$$c_{\vec{a}}(\vec{b}) = |\vec{b}| \cos\theta$$

- If $\theta < \frac{\pi}{2}$, then $c_{\vec{a}}(\vec{b}) > 0$
- If $\theta > \frac{\pi}{2}$, then $c_{\vec{a}}(\vec{b}) < 0$

Scalar Projection Using the Dot Product



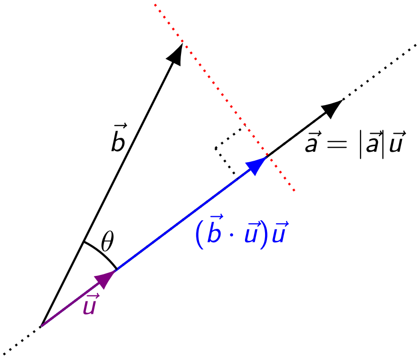
- Since $\vec{u}$ is orthogonal to $\vec{b} - p_{\vec{a}}(\vec{b})$,

$$\begin{aligned} 0 &= \vec{u} \cdot (\vec{b} - p_{\vec{a}}(\vec{b})) \\ &= \vec{u} \cdot \vec{b} - \vec{u} \cdot ((c_{\vec{a}}(\vec{b}))\vec{u}) \\ &= \vec{b} \cdot \vec{u} - c_{\vec{a}}(\vec{b}) \end{aligned}$$

- Therefore,

$$c_{\vec{a}}(\vec{b}) = \vec{b} \cdot \vec{u}$$

Trigonometric Formula for the Dot Product



- $\vec{b} \cdot \vec{u} = |\vec{b}| \cos\theta$
- Since $\vec{a} = |\vec{a}|\vec{u}$,

$$\begin{aligned} \vec{a} \cdot \vec{b} &= (|\vec{a}|\vec{u}) \cdot \vec{b} \\ &= |\vec{a}||\vec{b}| \cos\theta \end{aligned}$$

Standard Unit Vectors

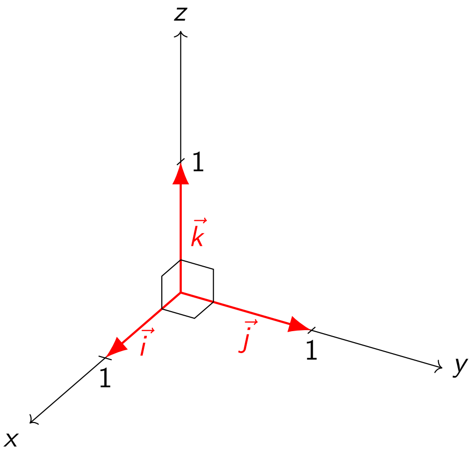
- The standard unit vectors point in the positive direction of each coordinate axis
- In 2-space

$$\vec{i} = \langle 1, 0 \rangle$$
$$\vec{j} = \langle 0, 1 \rangle$$

- In 3-space

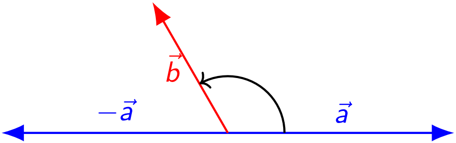
$$\vec{i} = \langle 1, 0, 0 \rangle$$
$$\vec{j} = \langle 0, 1, 0 \rangle$$
$$\vec{k} = \langle 0, 0, 1 \rangle$$

Standard Unit Vectors in 3-space

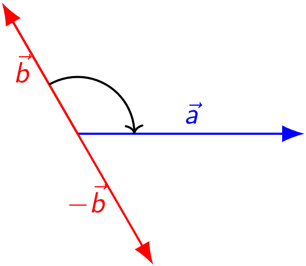


Orientation in 2-space

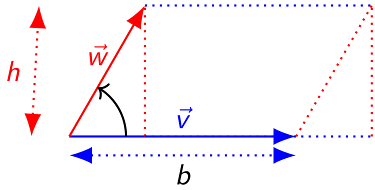
- An ordered pair of vectors $(\vec{a}, \vec{b})$ in 2-space has **positive orientation**, if $\vec{b}$ lies between $\vec{a}$ and $-\vec{a}$ going **counterclockwise** (from the x-axis toward the y-axis)



- $(\vec{b}, \vec{a})$ has negative orientation



Cross Product in 2-space



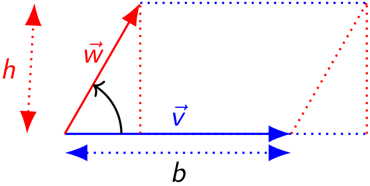
- The cross product $\vec{v} \times \vec{w}$ is defined to be the oriented area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$ (i.e., with vertices as $\vec{0}$, $\vec{v}$, $\vec{w}$, and $\vec{v} + \vec{w}$)
- Since $(\vec{v}, \vec{w})$ has positive orientation,

$$\vec{v} \times \vec{w} = hb$$

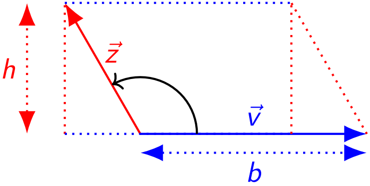
- Since $(\vec{w}, \vec{v})$ has negative orientation,

$$\vec{w} \times \vec{v} = -hb$$

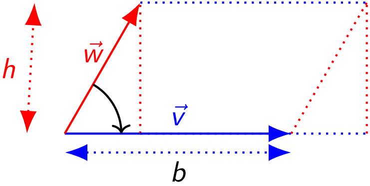
Cross Product = Oriented Area of a Parallelogram



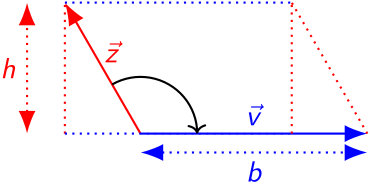
$\vec{v} \times \vec{w} = hb$



$\vec{v} \times \vec{z} = hb$

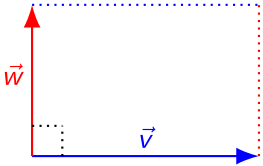


$\vec{w} \times \vec{v} = -hb$



$\vec{z} \times \vec{v} = -hb$

Special Cases

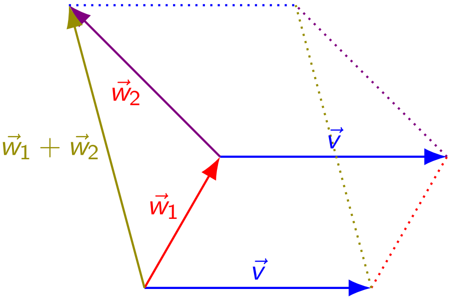


$$\vec{v} \cdot \vec{w} = 0$$

$$\vec{v} \times \vec{w} = |\vec{v}||\vec{w}|$$

$$\vec{v} \times \vec{z} = 0$$

The Cross Product is a Linear Function of Each Vector



$$\vec{v} \times (\vec{w}_1 + \vec{w}_2) = \vec{v} \times \vec{w}_1 + \vec{v} \times \vec{w}_2$$

Key Properties of Dot and Cross Products in 2-space

- Dot product is symmetric, positive definite, and bilinear
  - $\vec{v} \cdot \vec{w}$ is a scalar
  - $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
  - $\vec{v} \cdot \vec{v} = |\vec{v}|^2$
  - $\vec{v} \cdot (a\vec{w} + b\vec{z}) = a\vec{v} \cdot \vec{w} + b\vec{v} \cdot \vec{z}$
- Cross product is antisymmetric and bilinear
  - $\vec{v} \times \vec{w}$ is a scalar
  - $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$
  - $\vec{v} \times \vec{v} = 0$
  - $\vec{v} \times (a\vec{w} + b\vec{z}) = a(\vec{v} \times \vec{w}) + b(\vec{v} \times \vec{z})$

Calculating the Cross Product in 2-space

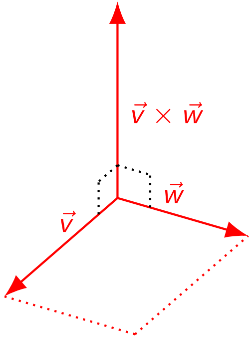
- Memorize the cross product of the standard unit vectors

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = 0$$
$$\vec{i} \times \vec{j} = 1, \ \vec{j} \times \vec{i} = -1$$

- Use asymmetric and bilinear properties of the cross product

$$\begin{aligned}(7\vec{i} - 11\vec{j}) \times (5\vec{i} + 3\vec{j}) &= 7(3)\vec{i} \times \vec{j} + (-11)5\vec{j} \times \vec{i} \\ &= 21 + 55 \\ &= 76\end{aligned}$$

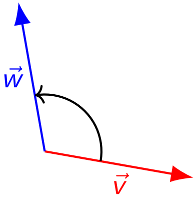
Cross Product in 3-space



- If $\vec{a}$ and $\vec{b}$ are vectors in 3-space, then their cross product $\vec{a} \times \vec{b}$ is a **vector**
- $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$
- The magnitude is equal to the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$
- The direction of $\vec{a} \times \vec{b}$ is given by the righthand rule

The Righthand Rule

- Suppose $\vec{v}$ and $\vec{w}$ are nonzero vectors in 3=space



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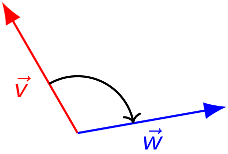
- If $\vec{w}$ lies less than 180 degrees counterclockwise of $\vec{v}$, then $\vec{v} \times \vec{w}$ points towards you



- If $\vec{w}$ lies less than 180 degrees clockwise of $\vec{v}$, then $\vec{v} \times \vec{w}$ points towards you
- Also, see youtube video on the righthand rule

Key Properties of Dot and Cross Products in 3-space

- Dot product is symmetric, positive definite, and bilinear
  - $\vec{v} \cdot \vec{w}$ is a scalar
  - $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
  - $\vec{v} \cdot \vec{v} = |\vec{v}|^2$
  - $\vec{v} \cdot (a\vec{w} + b\vec{z}) = a\vec{v} \cdot \vec{w} + b\vec{v} \cdot \vec{z}$
- Cross product is antisymmetric and bilinear
  - $\vec{v} \times \vec{w}$ is a *vector*
  - $\vec{v}$, $\vec{w}$, $\vec{v} \times \vec{w}$ obey the righthand rule
  - $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$
  - $\vec{v} \times \vec{v} = 0$
  - $\vec{v} \times (a\vec{w} + b\vec{z}) = a(\vec{v} \times \vec{w}) + b(\vec{v} \times \vec{z})$

Calculating the Cross Product in 3-space

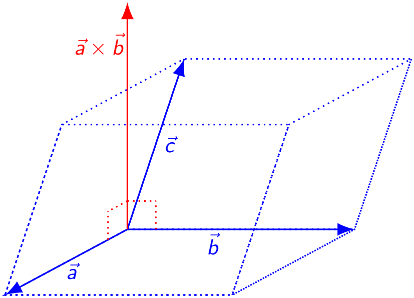
- Memorize the cross products for pairs of standard unit vectors

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$$
$$\vec{i} = \vec{j} \times \vec{k}, \ -\vec{i} = \vec{k} \times \vec{j}$$
$$\vec{j} = \vec{k} \times \vec{i}, \ -\vec{j} = \vec{i} \times \vec{k}$$
$$\vec{k} = \vec{i} \times \vec{j}, \ -\vec{k} = \vec{j} \times \vec{k}$$

- Use the antisymmetric and bilinear properties of the cross product

$$\begin{aligned}
&(\vec{i} + 2\vec{j} + 3\vec{k}) \times (5\vec{i} + 7\vec{j} + 11\vec{k}) \\
&= 2(11)\vec{j} \times \vec{k} + 3(7)\vec{k} \times \vec{j} \\
&\quad + 3(5)\vec{k} \times \vec{i} + 1(11)\vec{i} \times \vec{k} \\
&\quad + 1(7)\vec{i} \times \vec{j} + 2(5)\vec{j} \times \vec{i} \\
&= (22 - 21)\vec{i} + (15 - 11)\vec{j} + (7 - 10)\vec{k} \\
&= \vec{i} + 4\vec{j} - 3\vec{k}
\end{aligned}$$

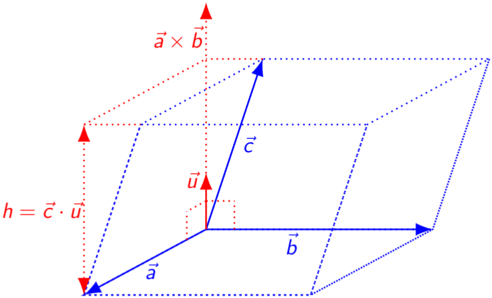
Parallelotope spanned by 3 Vectors in 3-space



- Three vectors $\vec{a}, \vec{b}, \vec{c}$ are **linearly independent**, if $\vec{c}$ does not lie in the plane containing $\vec{a}$ and $\vec{b}$
- Three linearly independent vectors span a parallelotope
- An ordered triple of linearly independent vectors, $(\vec{a}, \vec{b}, \vec{c})$, has positive orientation, if it obeys the righthand rule.
- $(\vec{a}, \vec{b}, \vec{c})$ has positive orientation if and only if

$$\vec{c} \cdot (\vec{a} \times \vec{b}) > 0.$$

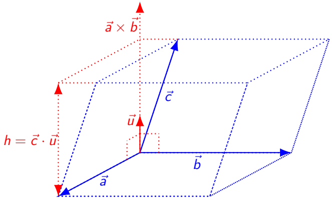
Volume of a Parallelotope



- Volume = (area of base)(height)
- Area of base $= |\vec{a} \times \vec{b}|$
- Height $= |c_{\vec{u}}(\vec{c})| = |\vec{c} \cdot \vec{u}|$, where

$$\vec{u} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

Unoriented Volume of a Parallelotope



The unoriented volume V of the parallelotope is equal to

$$\begin{aligned} |V| &= (\text{area of base})(\text{height}) \\ &= |\vec{a} \times \vec{b}||\vec{c} \cdot \vec{u}| \\ &= |\vec{a} \times \vec{b}| \left| \vec{c} \cdot \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} \right| \\ &= |(\vec{a} \times \vec{b}) \cdot \vec{c}| \end{aligned}$$

Oriented Volume of a Parallelotope

- Define the oriented volume of a parallelotope to be

$$V = (\vec{a} \times \vec{b}) \cdot \vec{c}.$$

- If $(\vec{a}, \vec{b}, \vec{c})$ has positive orientation, then $V > 0$
- If $(\vec{a}, \vec{b}, \vec{c})$ has negative orientation, then $V < 0$