MATH-UA 123 Calculus 3:
Dot Product and Orthogonal Projection

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START RECORDING

Point Versus Vector in Cartesian space

- Points
  - A point is a location in space
  - The coordinates of a point specifies its location relative to the origin
  - $P = (a, b, c)$ means the point located a units along the x-axis, b units along the y axis, and c units along the z axis **from the origin**
- Vectors
  - A vector is a *shift* in location, from one point to another
  - We draw a vector as an arrow
  - In coordinates $\vec{v} = \langle p, q, r \rangle$ means a change in x by p units, a change in y by q units, and a change in z by r units
- $P + \vec{v}$ means start at P and change the coordinates by $\vec{v}$.

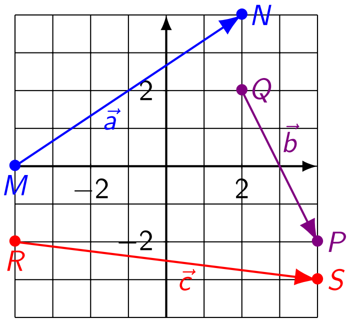
$$P + \vec{v} = (a, b, c) + \langle p, q, r \rangle = (a + p, b + q, c + r)$$

- Special case: Let $O = (0, 0, 0)$ be the origin, $P = (a, b, c)$ and $\vec{r} = \langle a, b, c \rangle$

$$P = (a, b, c) = O + \vec{r}$$

In this case, we call $\vec{r}$ the **position vector** of the point P.

Length of a Vector and Distance Between Two Points



By the Pythagorean Theorem,

$$(d(M,N))^2 = |N - M|^2 = |\vec{a}|^2 = 6^2 + 4^2 = 36 + 16 = 52$$

$$\implies \quad d(M,N) = \sqrt{52}$$

$$(d(P,Q))^2 = |P - Q|^2 = |\vec{b}|^2 = 2^2 + (-4)^2 = 4 + 16 = 20$$

$$\implies \quad d(P,Q) = \sqrt{20}$$

Length of a Vector and Distance Between Two Points in 2-space

- Length of a vector $\vec{v} = \langle v_1, v_2 \rangle$ is $|\vec{v}|$, where

$$|\vec{v}|^2 = (v_1)^2 + (v_2)^2.$$

- Distance between two points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ is the length of the vector from P to Q (or from Q to P):

$$\begin{aligned}(d(P,Q))^2 &= |Q - P|^2 \\ &= |\langle q_1 - p_1, q_2 - p_2 \rangle|^2 \\ &= (q_1 - p_1)^2 + (q_2 - p_2)^2\end{aligned}$$

Length and Distance in 3-space

- Length of a vector: If $\vec{v} = \langle v_1, v_2, v_3 \rangle$, then

$$|\vec{v}|^2 = |\langle v_1, v_2, v_3 \rangle|^2 = (v_1)^2 + (v_2)^2 + (v_3)^2$$

- Distance between two points: If $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$, then

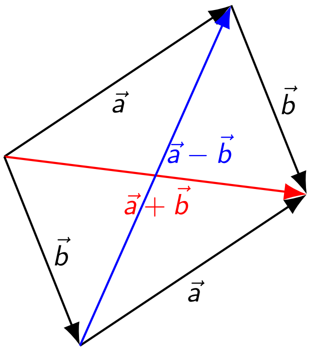
$$(d(P, Q))^2 = |Q - P|^2 = (q_1 - p_1)^2 + (q_2 - p_2)^2 + (q_3 - p_3)^2$$

Properties of the Length of a Vector

- If s is a scalar and $\vec{v}$ is a vector, then

$$|s\vec{v}| = |s||\vec{v}|$$

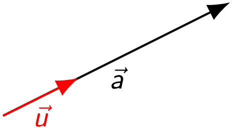
- Triangle inequality



$$|\vec{a} - \vec{b}| \leq |\vec{a}| + |\vec{b}|$$

$$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$$

Unit Vector

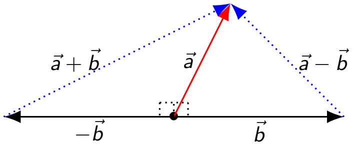


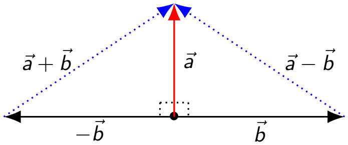
- A unit vector $\vec{u}$ a vector with length 1
- Any nonzero vector $\vec{a}$ can be rescaled into a unit vector $\vec{u}$ by dividing by its length

$$\vec{u} = \left(\frac{1}{|\vec{a}|}\right)\vec{a} = \frac{\vec{a}}{|\vec{a}|}$$

- Since the length of a unit vecctor is always 1, the only thing that can chage is its direction
- We will sometimes call a unit vector a direction

Orthogonal vectors

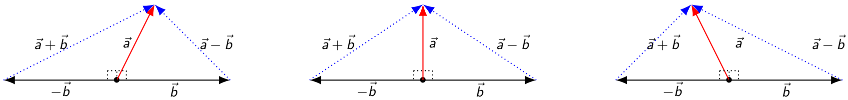




- A nonzero vector $\vec{a}$ is defined to be **orthogonal** or **perpendicular** or **normal** to a nonzero vector $\vec{b}$, if the angle from $\vec{a}$ to $-\vec{b}$ is equal to the angle from $\vec{b}$ to $\vec{a}$.
- Equivalently, by SAS and SSS,

$$|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|,$$

Acute, Right, Obtuse Angles



- If angle from $\vec{b}$ to $\vec{a}$ is 0, then

$$|\vec{a}-\vec{b}| = 0 \implies |\vec{a}-\vec{b}|^2 = 0$$

- If angle from $\vec{b}$ to $\vec{a}$ is acute, then

$$|\vec{a}-\vec{b}| < |\vec{a}+\vec{b}| \implies |\vec{a}+\vec{b}|^2 - |\vec{a}-\vec{b}|^2 > 0$$

- If angle from $\vec{b}$ to $\vec{a}$ is right, then

$$|\vec{a}-\vec{b}| = |\vec{a}+\vec{b}| \implies |\vec{a}+\vec{b}|^2 - |\vec{a}-\vec{b}|^2 = 0$$

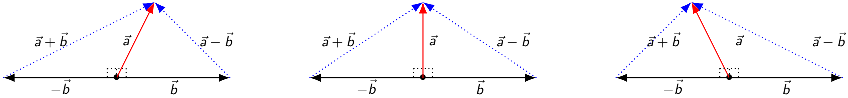
- If angle from $\vec{b}$ to $\vec{a}$ is obtuse, then

$$|\vec{a}-\vec{b}| > |\vec{a}+\vec{b}| \implies |\vec{a}+\vec{b}|^2 - |\vec{a}-\vec{b}|^2 < 0$$

- If angle from $\vec{b}$ to $\vec{a}$ is π radians, then

$$|\vec{a}+\vec{b}| = 0 \implies |\vec{a}+\vec{b}|^2 = 0$$

Dot Product



- A measure of the angle from a vector $\vec{b}$ to a vector $\vec{a}$ is

$$
\begin{aligned}
&|\vec{a}+\vec{b}|^2 - |\vec{a}-\vec{b}|^2 \\
&= |\langle a_1+b_1, a_2+b_2, a_3+b_3\rangle|^2 - |\langle a_1-b_1, a_2-b_2, a_3-b_3\rangle|^2 \\
&= (a_1+b_1)^2 + (a_2+b_2)^2 + (a_3+b_3)^2 \\
&\quad - ((a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2) \\
&= a_1^2 + 2a_1b_1 + b_1^2 + a_2^2 + 2a_2b_2 + b_2^2 + a_3^2 + 2a_3b_3 + b_3^2 \\
&\quad - (a_1^2 - 2a_1b_1 + b_1^2 + a_2^2 - 2a_2b_2 + b_2^2 + a_3^2 - 2a_3b_3 + b_3^2) \\
&= 4(a_1b_1 + a_2b_2 + a_3b_3)
\end{aligned}
$$

- An simple and important measure of angle between the vectors $\vec{a}$ and $\vec{b}$ is the dot product:

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

The Dot Product

- Define the **dot product** of two vectors $\vec{b} = \langle b_1, b_2, b_3 \rangle$ and $\vec{a} = \langle a_1, a_2, a_3 \rangle$ to be

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

- Geometric properties
  - As the angle from $\vec{b}$ to $\vec{a}$ increases from 0 to π radians, $\vec{a} \cdot \vec{b}$ decreases from $|\vec{a}||\vec{b}|$
  - $$\begin{aligned} \vec{a} \cdot \vec{b} &= |\vec{a}||\vec{b}|, \text{ if the angle is 0 radians} \\ \vec{a} \cdot \vec{b} &> 0, \text{ if the angle is acute} \\ \vec{a} \cdot \vec{b} &= 0, \text{ if the angle is } \frac{\pi}{2} \text{ radians} \\ \vec{a} \cdot \vec{b} &< 0, \text{ if the angle is obtuse} \\ \vec{a} \cdot \vec{b} &= -|\vec{a}||\vec{b}|, \text{ if the angle is } \pi \text{ radians} \end{aligned}$$
  - Special cases

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$
$$\vec{a} \cdot (-\vec{a}) = -|\vec{a}|^2$$

More Properties of the Dot Product

- Algebraic properties

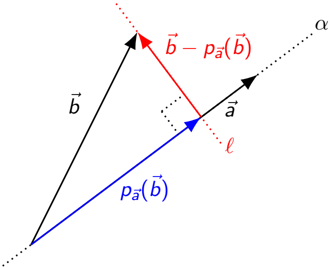
$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$
$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$
$$(c\vec{a}) \cdot \vec{b} = \vec{a} \cdot (c\vec{b}) = c(\vec{a} \cdot \vec{b})$$
$$(\vec{a}_1 + \vec{a}_2) \cdot \vec{b} = \vec{a}_1 \cdot \vec{b} + \vec{a}_2 \cdot \vec{b}$$
$$\vec{a} \cdot (\vec{b}_1 + \vec{b}_2) = \vec{a} \cdot \vec{b}_1 + \vec{a} \cdot \vec{b}_2$$

- Why use the dot product?
  - It is often (but not always) easier to do computations involving lengths of vectors and angles between vectors using the dot product than trigonometric formulas

Dot Product Grammar

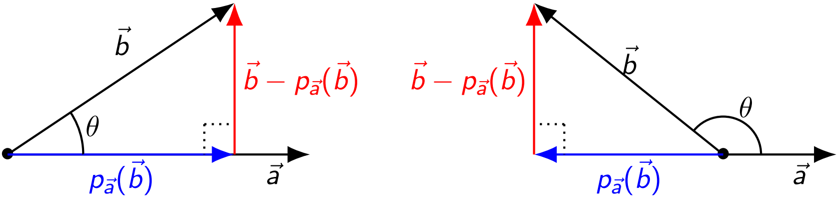
- Dot product
  - (vector)·(vector) = scalar
- Scalar multiplication
  - (scalar)(scalar) = scalar
  - (scalar)(vector) = (vector)(scalar) = vector
- Invalid
  - (scalar)·(scalar)
  - (scalar)·(vector)
  - (vector)(vector)
- $(2\vec{v}) \cdot (2\vec{w}) = 4(\vec{v} \cdot \vec{w}),$ and NOT $2(\vec{v} \cdot \vec{w})$
- $\langle a, b \rangle \cdot \langle v, w \rangle = av + bw$
- $(\vec{a} + \vec{b}) \cdot (\vec{v} + \vec{w}) \neq \vec{a} \cdot \vec{v} + \vec{b} \cdot \vec{w}$

Orthogonal Projection



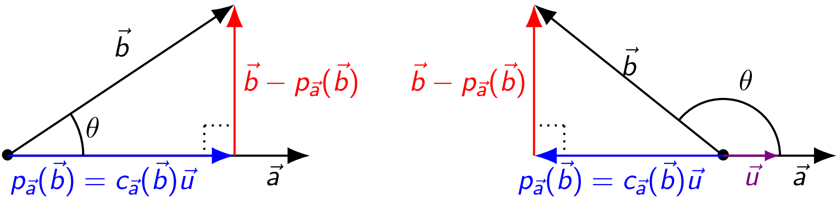
- Given nonzero vectors $\vec{a}$ and $\vec{b}$, there is a unique line ℓ that passes through the tip of $\vec{b}$ and is orthogonal to $\vec{a}$
- Let α be the line passing through $\vec{a}$
- The orthogonal projection $p_{\vec{a}}(\vec{b})$ is the vector that goes from the tail of $\vec{a}$ to the intersection of ℓ and α
- $p_{\vec{a}}(\vec{b})$ and $\vec{b} - p_{\vec{a}}(\vec{b})$ are orthogonal

Properties of Orthogonal Projection



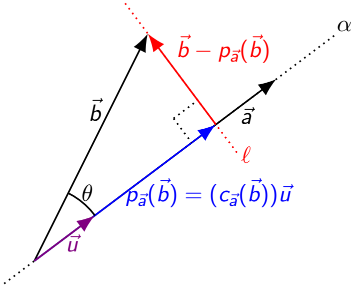
- If the angle between $\vec{a}$ and $\vec{b}$ is less than 90 degrees, then $p_{\vec{a}}(\vec{b})$ points in the same direction as $\vec{a}$.
- If the angle between $\vec{a}$ and $\vec{b}$ is greater than 90 degrees, then $p_{\vec{a}}(\vec{b})$ points in the opposite direction as $\vec{a}$.
- $0 \leq |p_{\vec{a}}(\vec{b})| \leq |\vec{b}|$
- $p_{\vec{a}}(\vec{b}) = \vec{0}$ if and only if $\vec{a}$ and $\vec{b}$ are orthogonal.
- $\vec{a}$ and $\vec{b}$ point in the same direction if and only if $p_{\vec{a}}(\vec{b}) = \vec{b}$
- $\vec{a}$ and $\vec{b}$ point in opposite directions if and only if $p_{\vec{a}}(\vec{b}) = -\vec{b}$

Scalar Projection



- Given a nonzero vector $\vec{a}$ and any vector $\vec{b}$, there is a scalar $c_{\vec{a}}(\vec{b})$ such that
$$p_{\vec{u}}(\vec{b}) = (c_{\vec{u}}(\vec{b}))\vec{u},$$
where $\vec{u}$ is the direction of $\vec{a}$.
- $c_{\vec{a}}(\vec{b})$ is called the **scalar projection** of $\vec{b}$ onto $\vec{a}$
- $c_{\vec{a}}(\vec{b}) = |\vec{b}| \cos\theta$

Orthogonal Projection Using the Dot Product



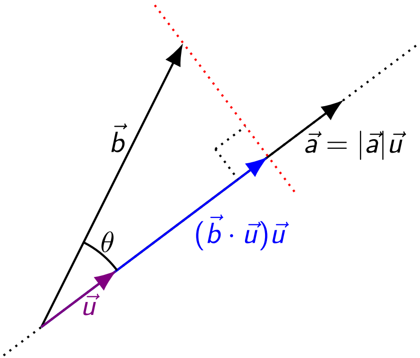
- Since $\vec{u}$ is orthogonal to $\vec{b} - p_{\vec{a}}(\vec{b})$,

$$\begin{aligned} 0 &= \vec{u} \cdot (\vec{b} - p_{\vec{a}}(\vec{b})) \\ &= \vec{u} \cdot \vec{b} - \vec{u} \cdot ((c_{\vec{a}}(\vec{b}))\vec{u}) \\ &= \vec{b} \cdot \vec{u} - c_{\vec{a}}(\vec{b}) \end{aligned}$$

- Therefore,

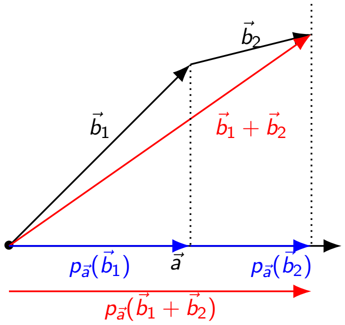
$$c_{\vec{a}}(\vec{b}) = \vec{b} \cdot \vec{u} \text{ and } p_{\vec{a}}(\vec{b}) = (\vec{b} \cdot \vec{u})\vec{u}$$

Trigonometric Formula for the Dot Product



$$\begin{aligned} \vec{a} \cdot \vec{b} &= |\vec{a}|\vec{u} \cdot \vec{b} \\ &= |\vec{a}||\vec{b}| \cos \theta \end{aligned}$$

Orthogonal Projection is Linear



- $p_{\vec{a}}(\vec{b}_1 + \vec{b}_2) = p_{\vec{a}}(\vec{b}_1) + p_{\vec{a}}(\vec{b}_2)$
- Proof using formulas

$$\begin{aligned} p_{\vec{a}}(\vec{b}_1 + \vec{b}_2) &= ((\vec{b}_1 + \vec{b}_2) \cdot \vec{u})\vec{u} \\ &= (\vec{b}_1 \cdot \vec{u} + \vec{b}_2 \cdot \vec{u})\vec{u} \\ &= (\vec{b}_1 \cdot \vec{u})\vec{u} + (\vec{b}_2 \cdot \vec{u})\vec{u} \\ &= p_{\vec{a}}(\vec{b}_1) + p_{\vec{a}}(\vec{b}_2) \end{aligned}$$

Properties of Orthogonal Projection

- Let s be a scalar and $\vec{a}, \vec{b}$ be nonzero vectors with directions
$$\vec{u} = \frac{\vec{a}}{|\vec{a}|}, \; \vec{v} = \frac{\vec{b}}{|\vec{b}|}$$
-
$$c_{\vec{a}}(\vec{b}) = c_{\vec{u}}(\vec{b}) = \vec{b} \cdot \vec{u}$$
$$p_{\vec{a}}(\vec{b}) = p_{\vec{u}}(\vec{b}) = (\vec{b} \cdot \vec{u})\vec{u}$$
-
$$c_{\vec{a}}(s\vec{b}) = sc_{\vec{a}}(\vec{b})$$
$$p_{\vec{a}}(s\vec{b}) = sp_{\vec{a}}(\vec{b})$$
-
$$c_{\vec{a}}(\vec{b}_1 + \vec{b}_2) = c_{\vec{a}}(\vec{b}_1) + c_{\vec{a}}(\vec{b}_2)$$
$$p_{\vec{a}}(\vec{b}_1 + \vec{b}_2) = p_{\vec{a}}(\vec{b}_1) + p_{\vec{a}}(\vec{b}_2)$$