# MATH-UA 123 Calculus 3: Points and Vectors

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START RECORDING

# Course Requirements: Quizzes and Exams

- Quizzes
  - Once or twice a week, available immediately after lecture
  - 15-30 minutes
  - Roughly 5 multiple choice or short answer questions
  - Minimal or no computations required
  - You have a two day window to take it at a time chosen by you
  - First quiz starts TODAY
- Exams
  - 2 midterms and a final
  - Details to be announced later
- Exam problems based on homework problems

# Course Requirements: Homework

- Written Homework
  - One per week
  - 5-10 problems
  - Distributed as a PDF on Gradescope every Wednesday
  - Due at midnight every Tuesday
  - You scan your solutions and upload them to Gradescope
  - Graded homework returned on Gradescope
- WebAssign Homework
  - Rolling assignment due dates
  - Keep an eye on the assignments to make sure you don't miss any
  - There are already assignments posted

# Office Hours

- Learn how to do assigned homework problems
- Learn how to figure out how to do problems
  - Important for doing well on exams
- Work on your homework during office hours
  - Get help when you get stuck, even on easy steps you're blanking out on
- Office hour regulars ALWAYS do better on exams and the course grade

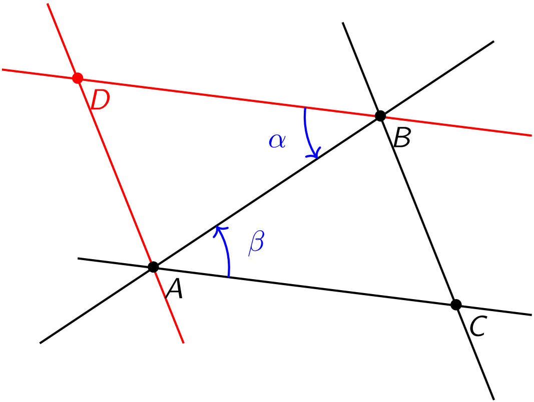
# Course Resources

- Lectures
  - Live lecture during class hours
  - Recording of lecture
  - Lecture slides
- Textbook
  - See calendar for sections covered
- IMPORTANT: Lectures alone will not cover all of the material
  - Textbook needed for material not covered in lecture

# Euclidean 2-Space

- Basic geometric objects: points, lines
- Basic geometric properties
  - Given two points, there is a unique line pssing through them
  - If two lines intersect, they intersect in a single point
  - Two lines that do not intersect are called parallel
  - Given a line and a point, there is a unique line containing the point and parallel to the given line
- Geometric measurements
  - Line segment between two points
    - Any two line segments can be compared in length
    - Any line segment can be rescaled in length by a factor of $n$
    - Any line segment can be subdivided into $n$ equal parts
- Angle between two line segments
  - Any two angles can be compared
  - Any angle can be rescaled by a factor of $n$
  - Any angle can be divided into 2 equal parts

# Points, Lines, Angles in 2-Space

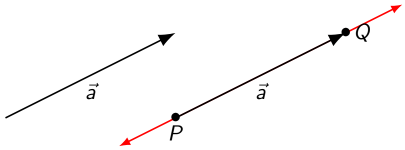



- Lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are parallel
- Lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ are parallel
- Line segments $\overline{AC}$ and $\overline{BD}$ have the same length
- Line segments $\overline{AD}$ and $\overline{BC}$ have the same length
- Angles $\alpha$ and $\beta$ are congruent

# Euclidean 3-space

- Basic geometric objects: points, lines, planes
- Basic geometric properties
  - Given two points, there is a unique line pssing through them
  - If two lines intersect, they intersect in a single point
  - If two lines do not intersect:
    - If they lie in a plane, then they are parallel
    - Otherwise, they are skew
  - Given a line and a point:
    - There is a unique plane containing them
    - There is a unique line containing the point and parallel to the given line
- Geometric measurements
- Line segments
  - Comparison of lengths
  - Rescaling and subdivision
- Angle between two line segments
  - Comparison of lengths
  - Rescaling by $n$ and subdivision by 2

# Vectors in 2-Space and 3-Space




- A vector $\vec{v}$ is an arrow with a given length and direction with no specific locatiopn in space
- Given two points $P$ and $Q$ in space, there is a unique vector $\vec{a}$, where, if the vector starts at $P$, then the vector ends at $Q$

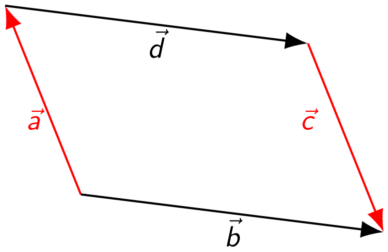
$$Q - P = \vec{a}$$

- Given a point $P$ and a vector $\vec{a}$, there is a unique point $Q$ such that $Q = P + \vec{a}$

$$Q = P + \vec{a}$$

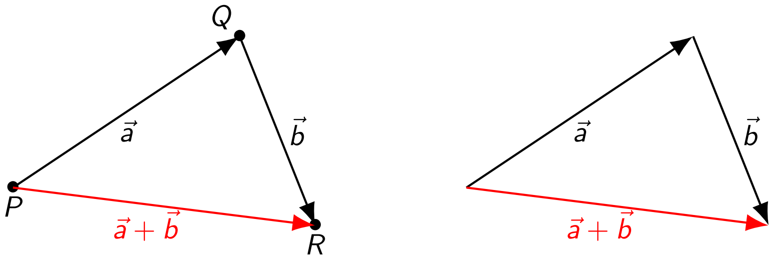
- $\vec{a} = Q - P \iff P = Q - \vec{a}$

# Comparison of Vectors



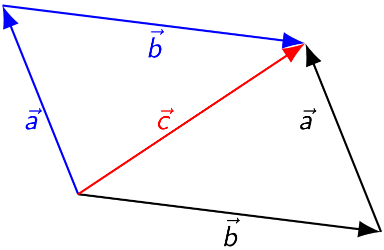

- $\vec{b} = \vec{d}$ because they have the same directions and lengths.
- $\vec{c} = -\vec{a}$ because $\vec{c}$ has the same length as $\vec{a}$ but its direction is the opposite of $\vec{a}$.
- Since opposite sides are pointing in either the same or opposite directions, they must be parallel. Therefore, this figure is a parallelogram.

# Addition of Two Vectors



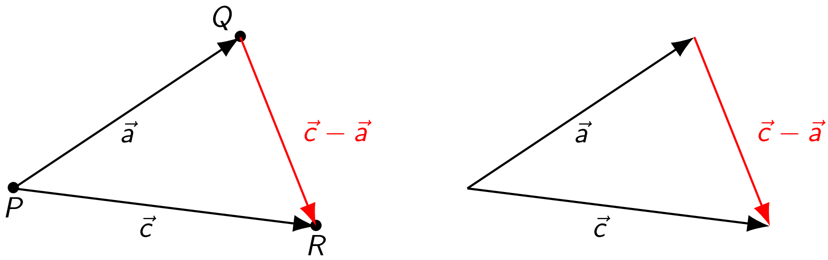

- Define the addition of two vectors by adding them one at a time.
- $P + (\vec{a} + \vec{b}) = (P + \vec{a}) + \vec{b} = Q + \vec{b} = R$

# Vector Addition is Commutative




$$\vec{c} = \vec{a} + \vec{b} = \vec{b} + \vec{a}$$

# Subtract Two Vectors



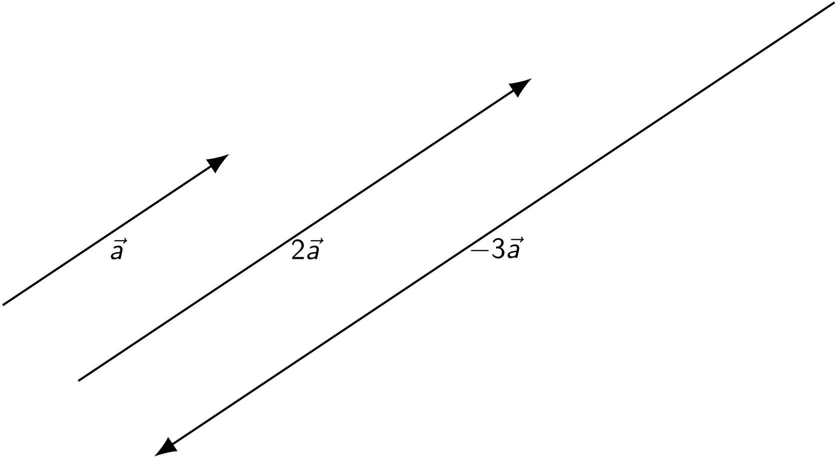

- Define $\vec{c}-\vec{a}$ to be the vector $\vec{b}$ such that

$$\vec{a}+\vec{b}=\vec{c}$$

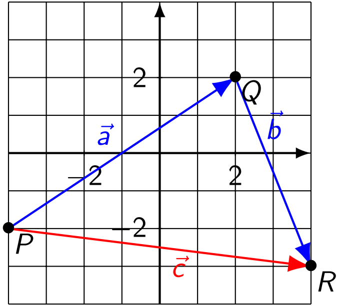
- In other words,

$$\vec{a}+(\vec{c}-\vec{a})=\vec{c}$$

# Scaling a Vector

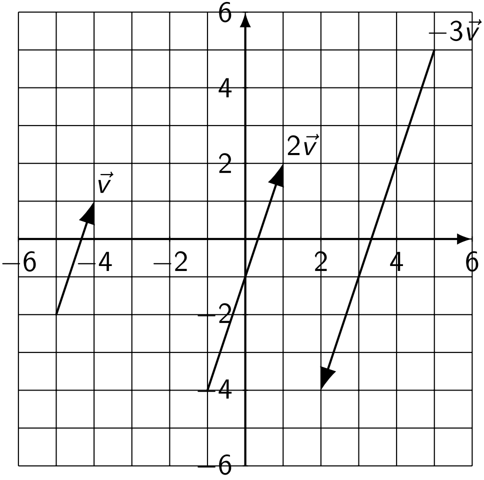


# Points and Vectors Using Coordinates



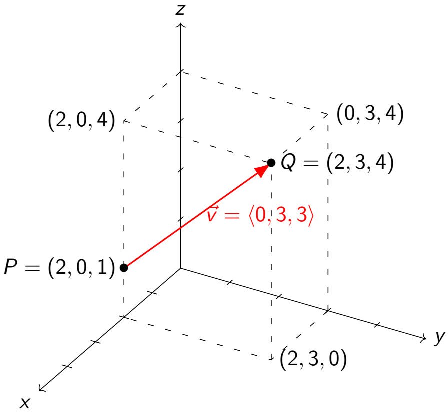

- $P = (-4, -2),\ Q = (2, 2),\ R = (4, -3)$
- $\vec{a} = Q - P = (2, 2) - (-4, -2) = \langle 6, 4 \rangle,\ Q = P + \vec{a}$
- $\vec{b} = R - Q = (4, -3) - (2, 2) = \langle 2, -5 \rangle,\ R = Q + \vec{b}$
- $\vec{c} = R - P = (4, -3) - (-4, -2) = \langle 8, -1 \rangle,\ R = P + \vec{c}$
- $\vec{a} + \vec{b} = \langle 6, 4 \rangle + \langle 2, -5 \rangle = \langle 8, -1 \rangle = \vec{c}$

# Rescaling a Vector




$$\vec{v} = \langle 1, 3 \rangle,\ 2\vec{v} = \langle 2, 6 \rangle,\ -3\vec{v} = \langle -3, -9 \rangle$$

# Point and vector in 3-space




- $\vec{v} = Q - P = (2, 3, 4) - (2, 0, 1) = \langle 0, 3, 3 \rangle$

# Valid operations for points, vectors, and scalars

$$
\begin{aligned}
\text{point} + \text{vector} &= \text{point}\\
\text{point} - \text{vector} &= \text{point}\\
\text{point} - \text{point} &= \text{vector}\\
\text{vector} + \text{vector} &= \text{vector}\\
\text{vector} - \text{vector} &= \text{vector}\\
(\text{scalar})(\text{vector}) &= (\text{vector})(\text{scalar}) = \text{vector}
\end{aligned}
$$

# Invalid operations

$$\text{point} + \text{point} = ?$$
$$(\text{point})(\text{point}) = ?$$
$$(\text{vector})(\text{vector}) = ?$$
$$\text{point} + \text{point} = ?$$
$$\text{scalar} + \text{point} = ?$$
$$\text{scalar} + \text{vector} = ?$$
$$(\text{scalar})(\text{point}) = ?$$

- **This is a good way to check your calculations**
- Mathematical grammar is critical for communicating clearly to both yourself and others
- In computer programming, this is called strong type checking

## Points and Vectors Using Coordinates

- Notation
  - A point is written using parentheses, such as $(-1, 3, 2)$
  - A vector is written using angle brackets, such as $\langle -1, 3, 2 \rangle$
- Addition and subtraction of points and vectors

$$(1,4,3) + (0,-1,2) \text{ is invalid}$$
$$(1,4,3) - (0,-1,2) = \langle 1,5,1 \rangle$$
$$\langle 1,4,3 \rangle + \langle 0,-1,2 \rangle = \langle 1,3,5 \rangle$$
$$\langle 1,4,3 \rangle - \langle 0,-1,2 \rangle = \langle 1,5,1 \rangle$$

- Rescaling

$$5(1,4,3) \text{ is invalid}$$
$$5\langle 1,4,3 \rangle = \langle 5,20,15 \rangle$$

# Properties of vector operations

- Vector addition is commutative and associative

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$
$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

- Scalar multiplication is associative

$$(st)\vec{v} = s(t\vec{v})$$

- Distributive property of scalar multiplication and vector addition

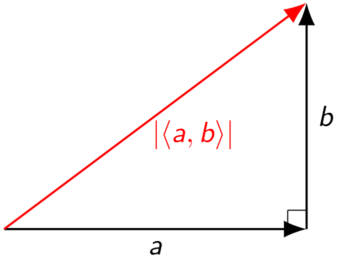
$$s(\vec{v} + \vec{w}) = s\vec{v} + s\vec{w}$$
$$(s + t)\vec{v} = s\vec{v} + t\vec{w}$$

# Length of a Vector

- Lengths of horizontal and vertical vectors

$$|\langle a, 0\rangle| = |a|$$
$$|\langle 0, b\rangle| = |b|$$

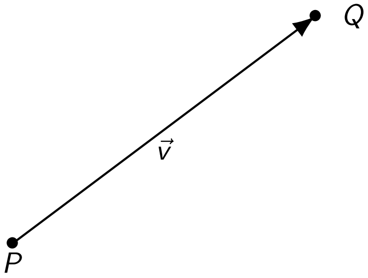
- Pythagorean Theorem $\implies$ length of vector




$$|\langle a, b\rangle|^2 = |\langle a, 0\rangle|^2 + |\langle 0, b\rangle|^2$$
$$= a^2 + b^2$$

# Distance Between Two Points

- Distance between two points = length of vector between them

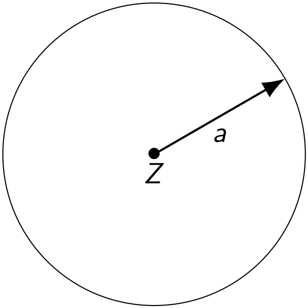



- $$d(P, Q) = |Q - P| = |\vec{v}|$$
- If $P = (p_1, p_2)$ and $Q = (q_1, q_2)$, then

$$(d(P, Q))^2 = |Q - P|^2 = |\langle q_1 - p_1, q_2 - p_2 \rangle|^2 = (q_1 - p_1)^2 + (q_2 - p_2)^2$$

# Circle

- A circle has a center (a point) and a radius (a positive scalar).
- The circle is the set of all points whose distance to the center is equal to the radius.
- Here is a circle $T$ with center at $Z$ and radius $a$



# Equation of Circle

- Suppose the center of the circle $T$ is $Z = (u, v)$ and its radius is $a$.

$$\begin{aligned} T &= \{Q \ : \ d(Z, Q) = a\} \\ &= \{(x, y) \ : \ d((u, v), (x, y)) = a\} \\ &= \{(x, y) \ : \ (x - u)^2 + (y - v)^2 = a^2\} \end{aligned}$$

- Suppose the center of the circle $C$ is $P = (x_0, y_0)$ and its radius is $r$.

$$\begin{aligned} C &= \{Q \ : \ d(P, Q) = r\} \\ &= \{(x, y) \ : \ d((x_0, y_0), (x, y)) = r\} \\ &= \{(x, y) \ : \ (x - x_0)^2 + (y - y_0)^2 = r^2\} \end{aligned}$$

# Circle centered at the origin

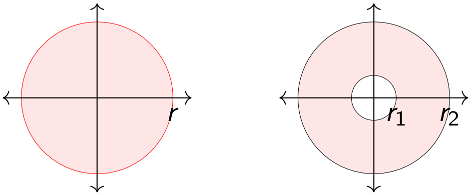
If the center of the circle is at the origin, then the equation becomes

$$d((x, y), (0, 0)) = r,$$

which is the same as

$$x^2 + y^2 = r^2.$$

# Disk and Annulus




- The disk of radius $r$ centered at $P = (x_0, y_0)$ is the set of all points whose distance to $P$ is less than or equal to $r$:

$$\begin{aligned} D &= \{Q \ : \ d(P,Q) \leq r\} \\ &= \{(x,y) \ : \ (x-x_0)^2 + (y-y_0)^2 \leq r^2 \end{aligned}$$

- The annulus with inner radius $r_1$ and outer radius $r_2$ centered at $P = (x_0, y_0)$ is:

$$\begin{aligned} A &= \{Q : \ r_1 \leq d(P,Q) \leq r_2\} \\ &= \{(x,y) \ : \ r_1^2 \leq (x-x_0)^2 + (y-y_0)^2 \leq r_2^2\} \end{aligned}$$

# Length and Distance in 3-space

- Length of a vector: If $\vec{v} = \langle a, b, c \rangle$, then

$$|\vec{v}|^2 = |\langle a, b, c \rangle|^2 = a^2 + b^2 + c^2$$

- Distance between two points: If $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$, then

$$(d(P, Q))^2 = |Q - P|^2 = (q_1 - p_1)^2 + (q_2 - p_2)^2 + (q_3 - p_3)^2$$

# Sphere, Ball, Spherical Shell in 3-Space

- A sphere $S$ in 3-space with center $P = (x_0, y_0, z_0)$ and radius $r > 0$ is defined to be

$$\begin{aligned} S &= \{Q \ : \ d(P,Q) = r\} \\ &= \{(x,y,z) \ : \ (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2\} \end{aligned}$$

- A ball $B$ in 3-space with center $P = (x_0, y_0, z_0)$ and radius $r > 0$ is defined to be

$$\begin{aligned} B &= \{Q \ : \ d(P,Q) \leq r\} \\ &= \{(x,y,z) \ : \ (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \leq r^2\} \end{aligned}$$

- A spherical shell $A$ in 3-space with center $P = (x_0, y_0, z_0)$, inner radius $r_1$ and outer radius $r_2$ is defined to be

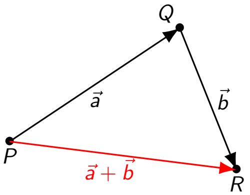
$$\begin{aligned} A &= \{Q \ : \ r_1 \leq d(P,Q) \leq r_2\} \\ &= \{(x,y,z) \ : \ r_1^2 \leq (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \leq r_2^2\} \end{aligned}$$

# Properties of Length and Distance

- If $s$ is a scalar and $\vec{v}$ is a vector, then

$$|s\vec{v}| = |s||\vec{v}|.$$

- Triangle inequality




- $|\vec{v}| \geq 0$, and $|\vec{v}| = 0 \iff \vec{v} = \vec{0}$
- $d(P,Q) \geq 0$, and $d(P,Q) = 0 \iff P = Q$
- $|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$, and $|\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}|$ *iff* $\vec{b} = \pm\vec{a}$
- $d(P,R) \leq d(P,Q) + d(Q,R)$, and $d(P,R) = d(P,Q) + d(Q,R)$ if and only if $P, Q, R$ lie on a line and $Q$ lies between $P$ and $R$