Midterm: Monday 10/25

Global Optimization

Focus on functions of 2 variables

D = domain in 2-space

f function on D

Goal: Find where the value of f is either maximal or minimal

If (x_0, y_0) is a point in D, such that $f(x_0, y_0) \leq f(x, y)$ for every $(x, y) \in D$

(x_0, y_0) is a minimum point (extremal, optimal)

$f(x_0, y_0)$ is the minimum value

Only one minimum value possible but more than one minimum is possible

Same story for max

D = all of 2-space

If f has a minimum, it is at a critical point

Strategy:

1) Find critical points

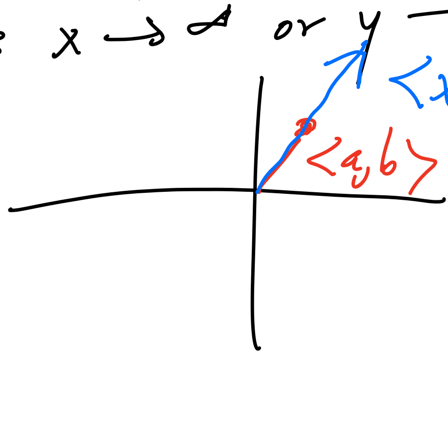
2) Look at what happens to f as $(x,y) \to$ infinity

3) Test critical points

Example

1) $f(x,y) = x^2 - 2xy + 3y^2$

As $x \to \infty$ or $y \to \infty$,



$\langle x,y \rangle = t\langle a,b \rangle$

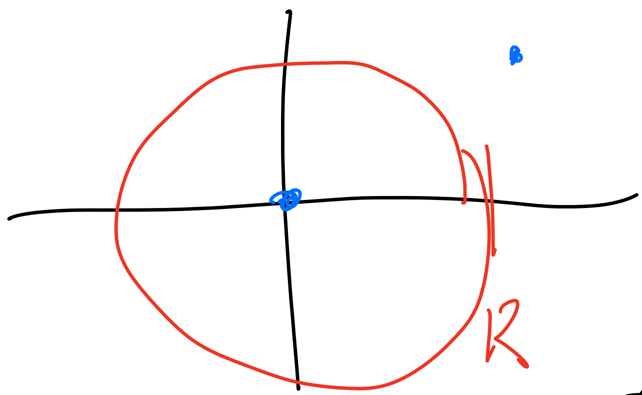
$t \to \infty$

$$f(x,y) = (x-y)^2 + 2y^2$$

$$x^2 - 2xy + y^2 + 2y^2$$

$f \to \infty$ as $(x,y) \to \infty$

$\Longrightarrow$ There has to be a minimum



$$f_x = 2x - 2y = 0 \Rightarrow y = x$$

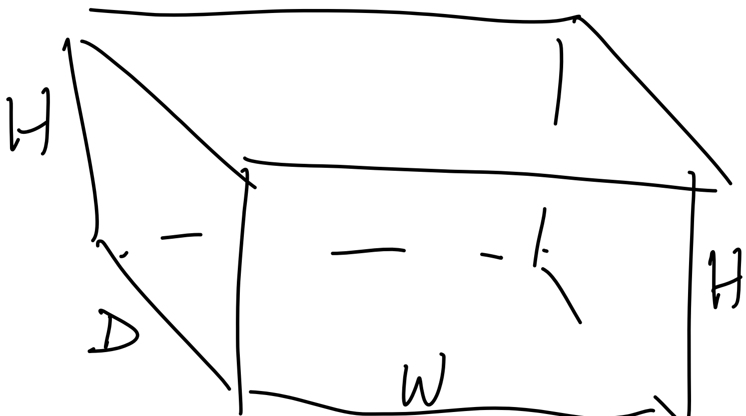
$$f_y = -2x + 6y = 0 \Rightarrow 4y = 0$$

$$\Rightarrow x = 0$$

$f(x,y) = y^2 - x^2$ ← saddle surface

$(0,0)$ critical point

but neither max nor min.



fixed

Cardboard box of volume V

No top

Use least amount of cardboard

Amount of cardboard measured by area

Find dimensions of best box

$$V = HWD$$

$$A = \text{area of bottom} + \text{area of sides}$$

$$= WD + 2HW + 2HD$$

$$H = \frac{V}{WD} \Rightarrow A = WD + 2\frac{V}{D} + 2\frac{V}{W}$$

$$A = 2V\left(\frac{1}{D}+\frac{1}{W}\right) + WD$$

$$A(W,D) \qquad (W,D):\ W>0,\ D>0$$

has to be analyzed carefully

If D or $W \to \infty$, last term $\to \infty$

If D or $W \to 0$, first term $\to \infty$

$\Longrightarrow$ minimum exists and is at a critical point.

$$A_W = -\frac{2V}{W^2} + D = 0$$

$$A_D = \frac{-2V}{D^2} + W = 0$$

$$D = \frac{2V}{W^2} \qquad W = \frac{2V}{D^2} = 2V\left(\frac{W^4}{4V^2}\right)$$

$$\frac{w^4}{2V^2} = w \Rightarrow \frac{w^4}{2V^2} - w = 0$$

$$\Rightarrow w\left(\frac{w^3}{2V^2} - 1\right) = 0$$

$$\Rightarrow \cancel{w = 0} \quad \text{or} \quad \frac{w^3}{2V^2} = 1$$

$$\Rightarrow w^3 = 2V^2 \Rightarrow w = \sqrt[3]{2V^2}$$

$$\Rightarrow D = \frac{2V}{(2V^2)^{2/3}}$$

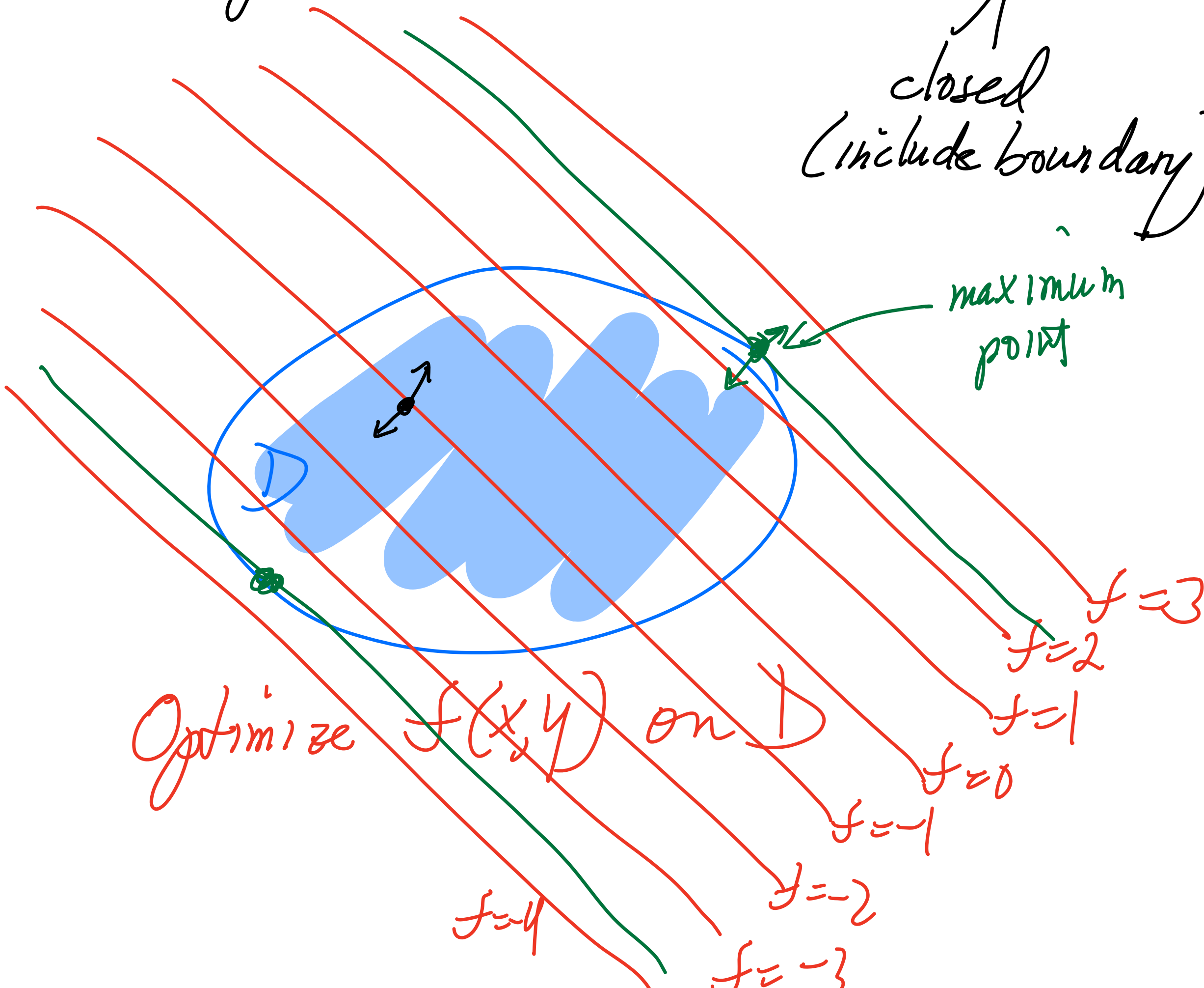
$$H = \frac{V}{wD}$$

etc.

Note: Did not use 2nd derivative test

Not needed.

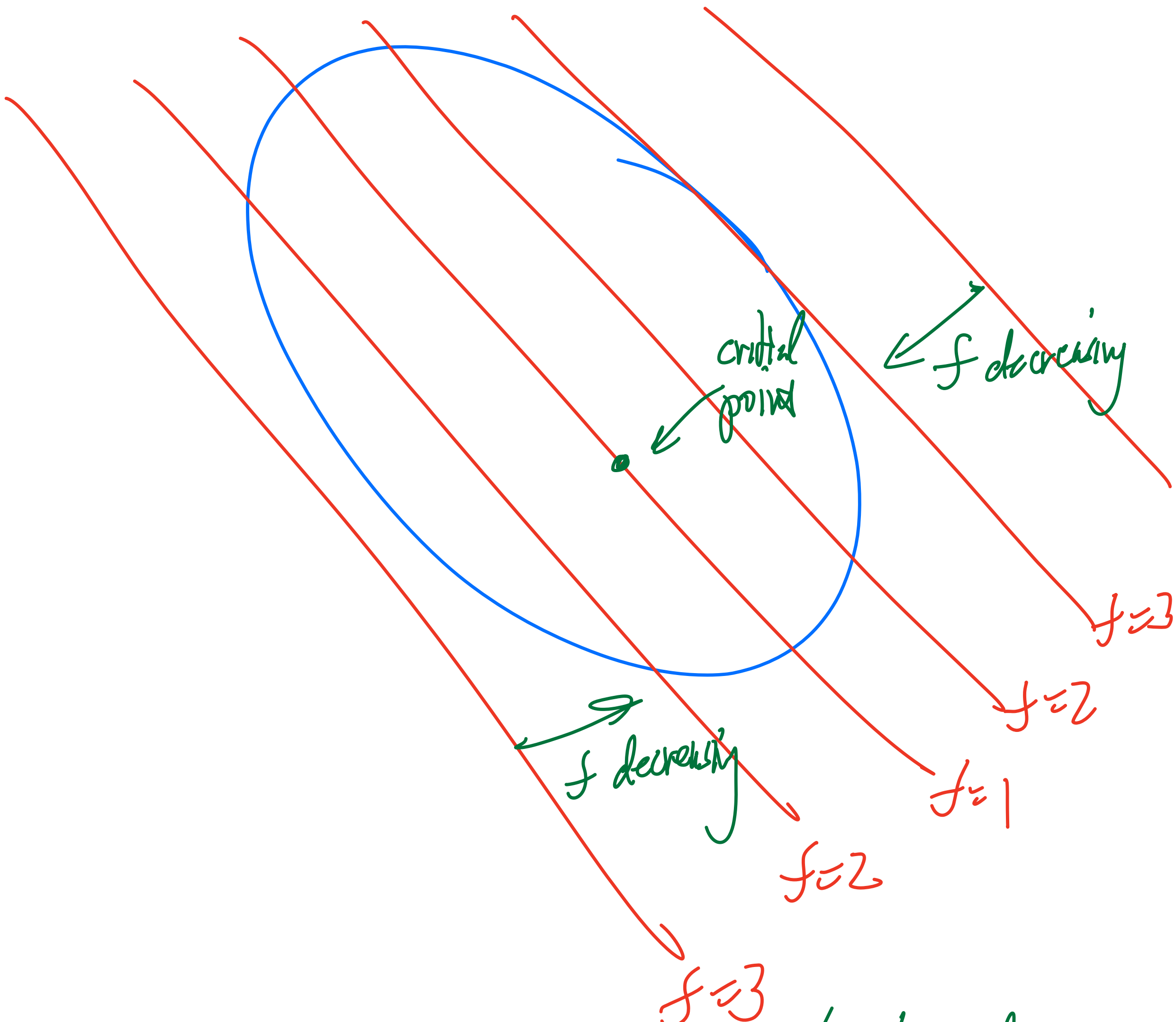
Optimization on a bounded, closed (include boundary) domain



Optimize $f(x,y)$ on D

For this example, optima cannot be in the interior of D

$$-4 < f(x,y) < 3 \,, \text{ if } (x,y) \in D$$

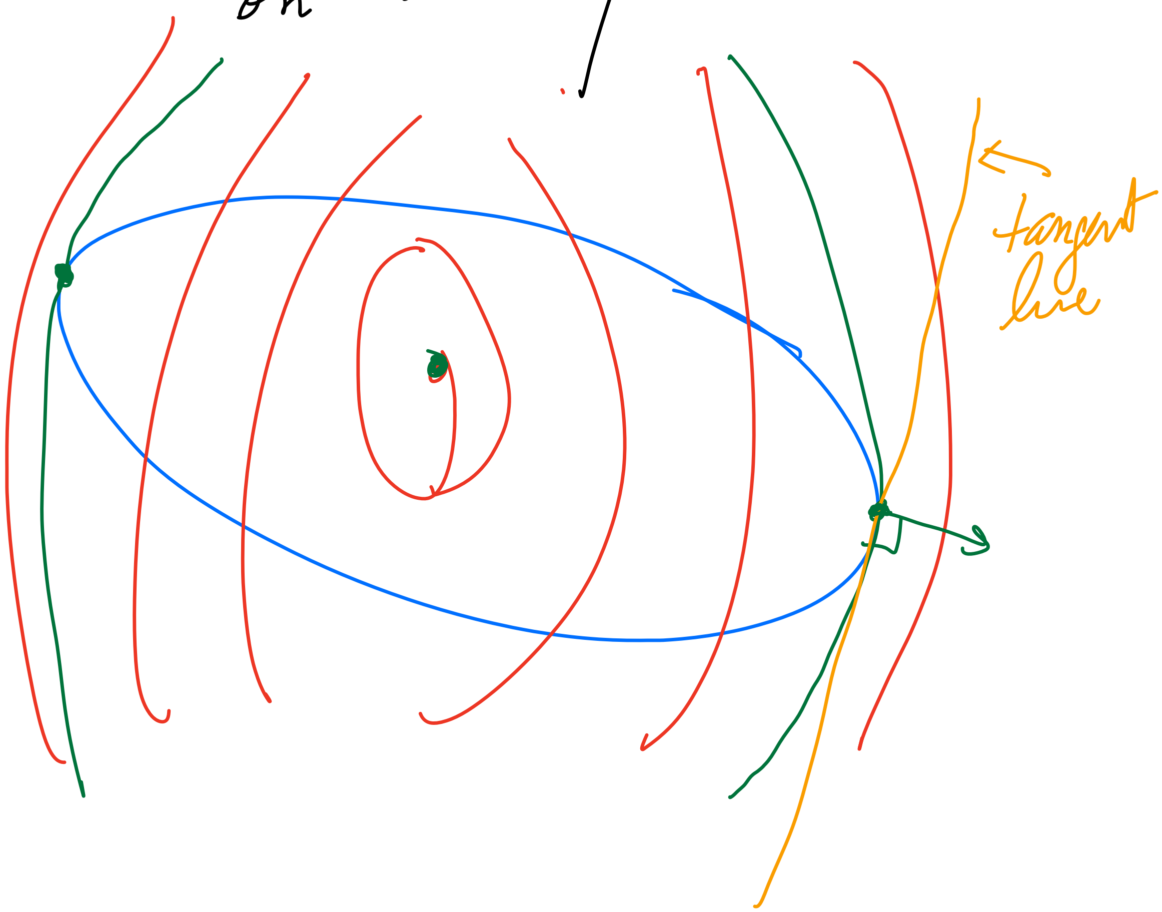


Here, maximum is on the boundary but minimum is at a critical point in the interior of D

Strategy

1) Find critical points of that are in D
(Remember to check if a critical point is in D or not)

2) Find potential max or min on boundary



i.e, points where the contour and the boundary of D are tangent (i.e, they have the same tangent line)

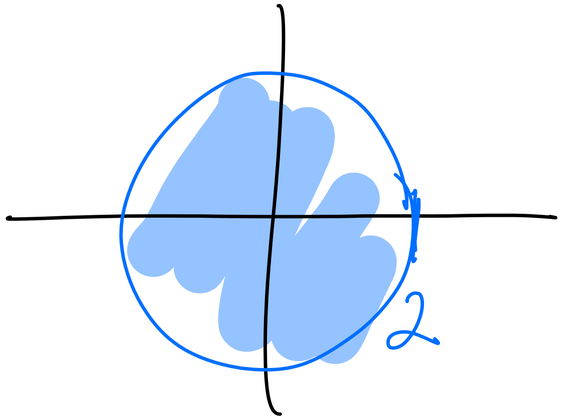
3) Test the value of f at each point found in 1) and 2 to identify optimal values and where they occur.

To do this using computations, need a way to describe D using an equation

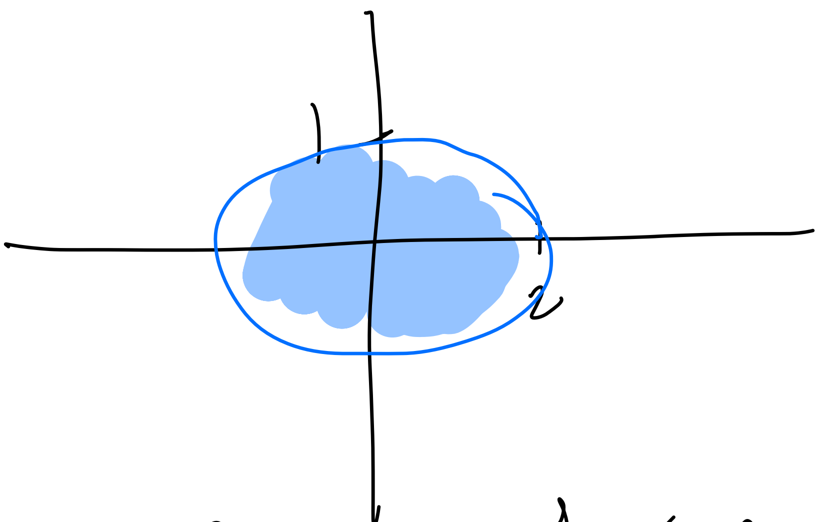
$$D = \{ g(x,y) \leq c \}$$

example

$$D = \{ x^2 + y^2 \leq 4 \}$$



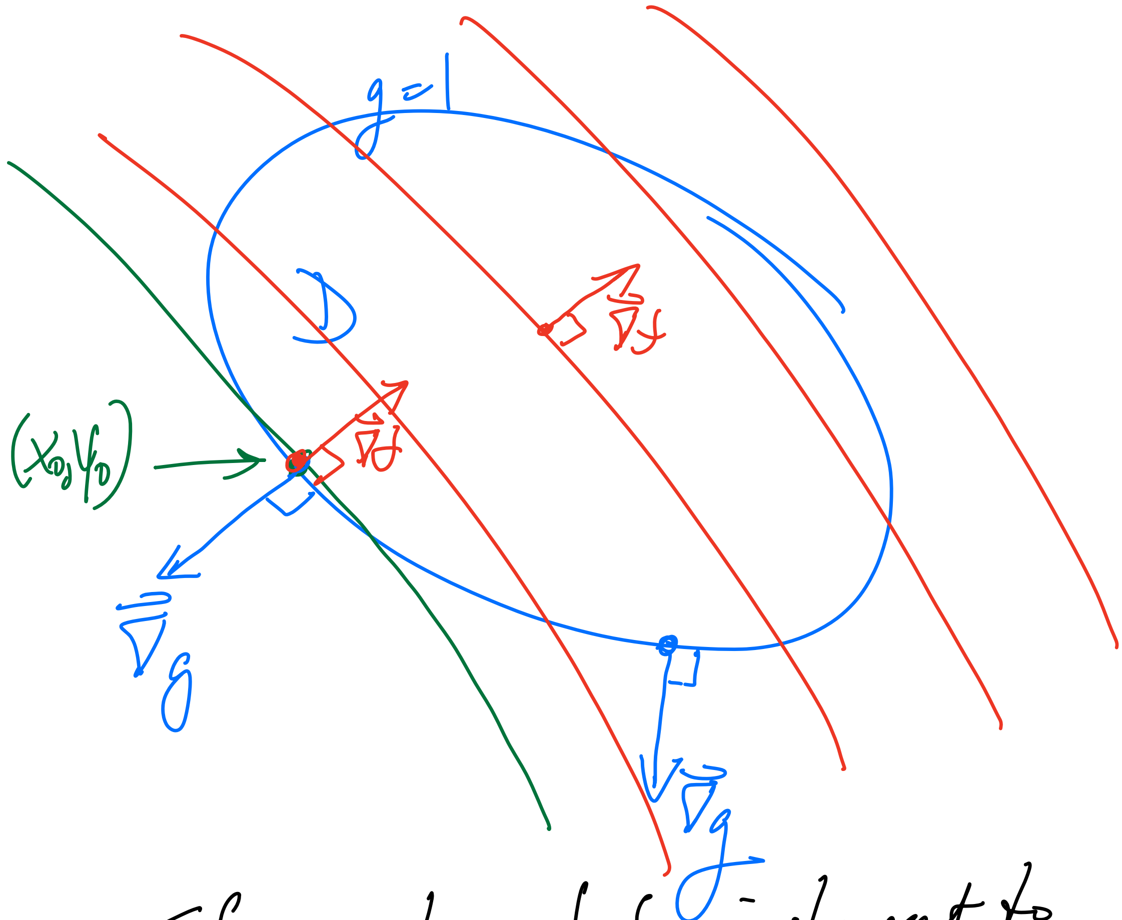
$$D = \left\{ \frac{x^2}{4} + y^2 \le 1 \right\}$$



So if $D = \{ g(x,y) \le c \}$

boundary of $D = \{ g(x,y) = c \}$

so boundary of D is a contour of g



If contour of f is tangent to contour of g at a point (x_0, y_0)

$\Longrightarrow$ $\vec{\nabla} f(x_0, y_0)$, $\vec{\nabla} g(x_0, y_0)$ have same or opposite directions

Assume: $\vec{\nabla} g(x,y) \neq 0$ for every (x,y) in boundary of D

Then at (x_0, y_0), there is a scalar λ such that

$$\boxed{\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0)}$$

Strategy, version 2

1) Find critical points of f that are in D

2) Find all points (x_0, y_0) on the boundary where there is a scalar λ such that

$$\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0)$$

(critical point: $\vec{\nabla} f(x_0, y_0) = \vec{0}$)

Such points are potential max or min points

3) Test f at points found.

Lagrange multipliers

λ called a Lagrange multiplier

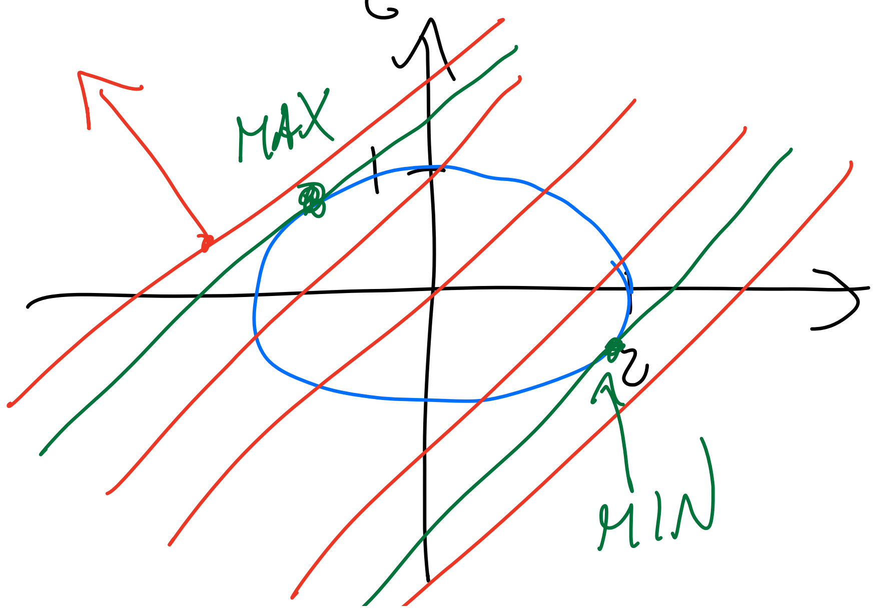
Example

$f = c \iff y = x + c$

Optimize $f(x,y) = y - x$

on the domain

$$D = \left\{ \frac{x^2}{4} + y^2 \leq 1 \right\} = \{ g \leq 1 \}$$



Here, $g(x,y) = \frac{x^2}{4} + y^2$

$$\vec{\nabla} g = \left\langle \frac{x}{2}, 2y \right\rangle \neq \vec{0}$$

if $(x,y) \neq (0,0)$

$f(x,y) = y - x$

$$\vec{\nabla} f = \langle -1, 1 \rangle \neq \vec{0}$$

$\Rightarrow$ No critical points

$\Rightarrow$ Optimal points are all on boundary

Find (x_0, y_0) and λ such that

$$\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0)$$

$$\langle -1, 1 \rangle = \lambda \left\langle \frac{x}{2}, 2y \right\rangle$$

First, note $\lambda \neq 0$

$$-1 = \lambda \frac{x}{2}$$

$$1 = \lambda 2y$$

Common strategy: If you know $\lambda \neq 0$, then solve each equation for λ^{-1}

$$\lambda^{-1} = -\frac{x}{2}$$

$$\lambda^{-1} = 2y$$

$$\Rightarrow \quad -\frac{x}{2} = 2y \Rightarrow \boxed{y = -\frac{1}{4}x}$$

but also, $g(x,y) = 1$

$$\frac{x^2}{4} + y^2 = 1$$

$$\frac{x^2}{4} + \left(-\frac{x}{4}\right)^2 = 1$$

$$\frac{x^2}{4} + \frac{x^2}{16} = 1 \Rightarrow \frac{5}{16}x^2 = 1$$

$$x = \frac{\pm 4}{\sqrt{5}}$$

So either $x = \frac{4}{\sqrt{5}}$ and $y = -\frac{1}{4}x = \frac{-1}{\sqrt{5}}$

or $x = \frac{-4}{\sqrt{5}}$ and $y = -\frac{1}{4}x = \frac{1}{\sqrt{5}}$

$\Rightarrow$ 2 possible optimal points

$$\left(\frac{4}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right), \left(\frac{-4}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

Common mistake

$x = \pm\frac{4}{\sqrt{5}}$ and $y = -\frac{1}{4}x = -\frac{1}{4}\left(\pm\frac{4}{\sqrt{5}}\right)$

$$= \mp\frac{1}{\sqrt{5}}$$

$$\Rightarrow \left(\frac{4}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(\frac{4}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right), \left(\frac{-4}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right),$$

$$\left(\frac{-4}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$$

$$f\left(\frac{4}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right) = y - x$$

Min

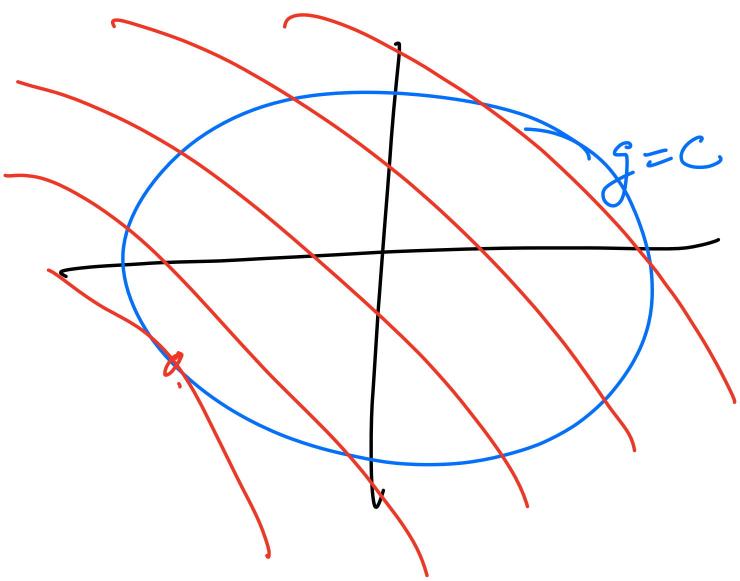
$$= \frac{-1}{\sqrt{5}} - \frac{4}{\sqrt{5}} = \frac{-5}{\sqrt{5}}$$

$$= -\sqrt{5}$$

$$f\left(\frac{-4}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}} - \frac{-4}{\sqrt{5}} = \sqrt{5}$$

MAX

Lagrange multiplier is for optimizing a function f restricted to a contour $\{g = c\}$



Terminology

f = objective function

$g = c$ is constraint equation

Goal: Constrained optimization

Simplifying assumptions

1) Contour $g = c$ is bounded

2) $\vec{\nabla} g \neq 0$ along contour (x_0, y_0)

Optimal points will occur at points where

$$\left.\begin{aligned} \vec{\nabla} f(x_0, y_0) &= \lambda \vec{\nabla} g(x_0, y_0) \\ &\text{for some scalar } \lambda \\ &\text{AND} \\ g(x_0, y_0) &= c \end{aligned}\right\} \begin{aligned} &3 \text{ equations} \\ &3 \text{ unknowns} \\ &\lambda, x_0, y_0 \end{aligned}$$

2 approaches to solving these equations

1) a) Solve for λ (or λ^{-1}) in each of 1^{st} two equations

b) Combine to get equation in x, y only

c) Get 2 equations for (x, y).

d) Solve them for (x, y)

2) a) Use 1^{st} two equations to solve for x and y in terms of λ only

b) Plug these formulas into third equation and solve for λ

c) Use equations from a) to find (x, y)

Example

$$f(x,y) = (x+2)^2 + (y-1)^2$$

Constraint $\quad \underbrace{x^2+y^2}_{g(x,y)} = 1$

$$\vec{\nabla} f = \langle 2(x+2), 2(y-1) \rangle$$

$$\vec{\nabla} g = \langle 2x, 2y \rangle \neq 0$$

on contour $g=1$

Equations

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

$$g = 1$$

$$\langle 2(x+2), 2(y-1) \rangle = \lambda \langle 2x, 2y \rangle$$
$$x^2 + y^2 = 1$$

$$2(x+2) = \lambda(2x)$$
$$\Rightarrow \quad 2(y-1) = \lambda(2y)$$
$$x^2 + y^2 = 1$$

$$x + 2 = \lambda x$$
$$\Rightarrow \quad y - 1 = \lambda y$$
$$x^2 + y^2 = 1$$

Approach 1

1) Solve for λ

$$\left.\begin{aligned} \lambda &= \frac{x+2}{x} \\ \lambda &= \frac{y-1}{y} \end{aligned}\right\}$$ Bad if $x=0$ or $y=0$

2) Check original equations

If $x=0$, first equation becomes

$$2=0$$

$$\Rightarrow x \neq 0$$

If $y=0$, 2^{nd} equation becomes

$$-1=0$$

$$\Rightarrow y \neq 0$$

3) $$\frac{x+2}{x} = \frac{y-1}{y}$$

$$1+\frac{2}{x}=1-\frac{1}{y}$$

$$\frac{2}{x}=-\frac{1}{y}$$

$$2y=-x \Rightarrow y=-\frac{x}{2}$$

4) $x^2+y^2=1$

$$\Rightarrow x^2+\frac{x^2}{4}=1 \Rightarrow \frac{5}{4}x^2=1$$

$$\Rightarrow x=\frac{\pm 2}{\sqrt{5}}$$

$$\Rightarrow x=\frac{2}{\sqrt{5}},\ y=-\frac{1}{2}\left(\frac{2}{\sqrt{5}}\right)$$

or

$$x=\frac{-2}{\sqrt{5}},\ y=-\frac{1}{2}\left(\frac{-2}{\sqrt{5}}\right)$$

Approach 2

$$x + 2 = \lambda x$$
$$y - 1 = \lambda y$$
$$x^2 + y^2 = 1$$

1) Solve for x, y in terms of λ:

$$x - \lambda x = -2$$
$$\boxed{x = \frac{-2}{1-\lambda} = \frac{2}{\lambda - 1}}$$
$$y - \lambda y = 1$$
$$\boxed{y = \frac{1}{1-\lambda} = \frac{-1}{\lambda - 1}}$$

2) Plug into constraint equation

$$\left(\frac{2}{\lambda-1}\right)^2 + \left(\frac{-1}{\lambda-1}\right)^2 = 1$$

$$\frac{5}{(\lambda-1)^2} = 1$$

$$(\lambda-1)^2 = 5$$

$$\lambda - 1 = \pm\sqrt{5}$$

$$\lambda = 1 \pm \sqrt{5}$$

$$\lambda - 1 = \pm\sqrt{5}$$

$$x = \frac{2}{\lambda-1},\ y = -\frac{1}{\lambda-1}$$

So

$$x = \frac{2}{\sqrt{5}},\ y = -\frac{1}{\sqrt{5}}$$

or

$$x = \frac{-2}{\sqrt{5}},\ y = \frac{1}{\sqrt{5}}$$

Last step: Calculate f
at the points found.