Midterm: Monday 10/25
- Covers everything up to 10/18
  (up to Lagrange multipliers)
- HW solutions will be available next week

Quiz 8, #5

Tangent plane to $f(x,y) = x^2 y$ at $(1,2)$ is

$$z = 2 + 2xy(x-1) + x^2(y-2)$$

True or false?

FALSE:

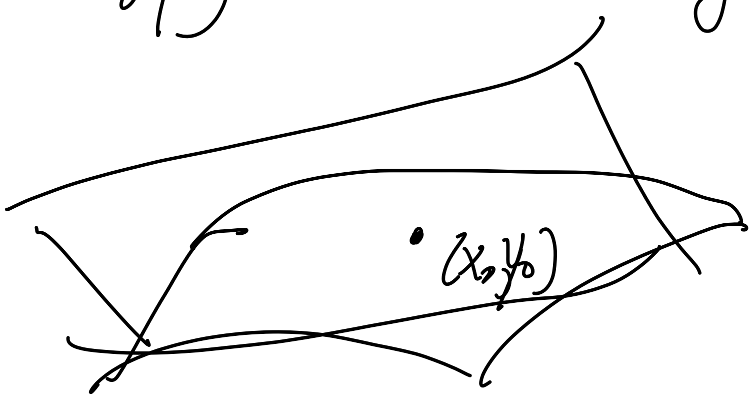
Equation of a plane is linear

$$z = ax + by + c$$

where a, b, c are constants (no x, y, z in them)

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

(x_0,y_0) is where the tangent plane is



In problem, $(x_0,y_0)=(1,2)$

In formula, use $f_x(1,2)$, $f_y(1,2)$ instead of $f_x(x,y)$, $f_y(x,y)$

Tangent plane to contour

$$f(x,y,z) = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16}$$

Tangent plane to contour $f(x,y,z)=3$ at $(2,3,4)$: $f(2,3,4)=3$

Idea: Use implicit differentiation
Use differentials

$f(x, y, z) = 3$

means $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 3$

$$\frac{x\,dx}{2} + \frac{2y\,dy}{9} + \frac{2z\,dz}{16} = 0$$

At $(2, 3, 4)$
$x \;\; y \;\; z$

$$dx + \frac{2}{3}dy + \frac{1}{2}dz = 0$$

$$\boxed{\begin{aligned} dx &\cong x - 2 \\ dy &\cong y - 3 \\ dz &\cong z - 4 \end{aligned}}$$ Plug in

Get a linear equation that approximates contour at $(2,1,3)$

Linear equation defines tangent plane at $(2,1,3)$

HW5 #5 $g=0 \Rightarrow dg=0$

$dg = g_x\,dx + g_y\,dy + g_z\,dz = 0$

$dx \overset{\sim}{=}$

$dy \overset{\sim}{=}$

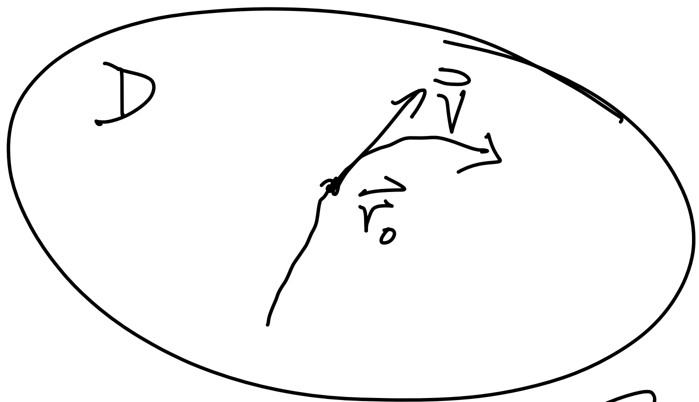
$dz \overset{\sim}{=}$

Plug $(x,y,z)=(2,1,7)$ into

g_x, g_y, g_z

Use g_x, g_y, g_z only at $(2,1,7)$

Directional derivative



$$f : D \longrightarrow \mathbb{R}$$

Define $D_{\vec{v}} f(\vec{r}_0)$:

1) Choose a parameterized curve $\vec{r}(t)$ where $\vec{r}(0) = \vec{r}_0$, $\vec{r}\,'(0) = \vec{v}$

$$D_{\vec{v}} f(\vec{r}_0) = \left.\frac{d}{dt}\right|_{t=0} f(\vec{r}(t))$$

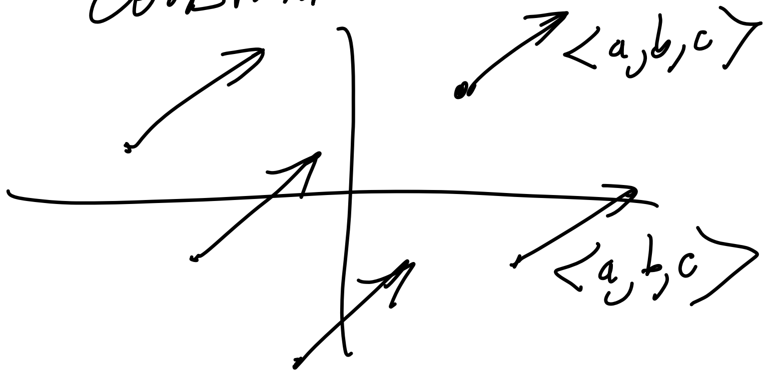
function of t only

2) $D_{\vec{v}} f(\vec{r}_0) = \vec{v} \cdot \vec{\nabla} f(\vec{r}_0)$

where $\vec{\nabla} f(\vec{r}) = \langle f_x, f_y, f_z \rangle$
which is a vector field
called the gradient of f

Examples

i) Linear function
$f(x,y,z) = ax + by + cz + d$
a, b, c constants
$\vec{\nabla} f(x,y,z) = \langle a, b, c \rangle$
constant vector field



2) $f(\vec{r}) = |\vec{r}|^2$

$$f(x,y,z) = x^2+y^2+z^2$$

$$\vec{\nabla} f = \langle 2x, 2y, 2z \rangle$$

$$\vec{\nabla} f = 2\vec{r}$$

Given $\vec{u}$,

$$D_{\vec{u}} f = \vec{u} \cdot \vec{\nabla} f$$

$$D_{\vec{u}} f = \vec{u} \cdot (2\vec{r})$$

$$D_{\vec{u}} f = 2\vec{u} \cdot \vec{r}$$

3) $f(\vec{r}) = x^3+y^3+z^3$

$$\vec{\nabla} f = \langle 3x^2, 3y^2, 3z^2 \rangle$$

$$= 3\langle x^2, y^2, z^2 \rangle$$

$$D_{\vec{u}} f = \vec{u} \cdot \vec{\nabla} f$$

$= \dots$ (no simple formula like)

$$\vec{u} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right\rangle$$

$$D_{\vec{u}} f(1,-1,1) = \vec{u} \cdot \vec{\nabla} f(\overset{x}{\overset{\parallel}{1}}, \overset{y}{\overset{\parallel}{-1}}, \overset{z}{\overset{\parallel}{1}})$$

$$= \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right\rangle \cdot \langle 3, 3, 3 \rangle$$

$$= \sqrt{3} + \sqrt{3} - \sqrt{3} = \sqrt{3}$$

Output of f is scalar

Output of $\vec{\nabla} f$ is vector

Output of $D_{\vec{u}} f$ is scalar

$$\overset{\parallel}{\vec{u} \cdot \vec{\nabla} f}$$

Directional derivative $D_{\vec{u}} f$

where $\vec{u}$ is unit

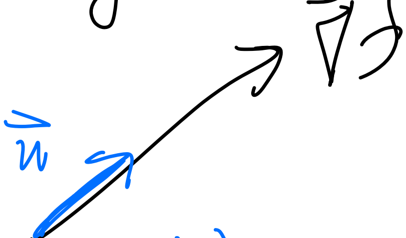
$$D_{\vec{u}} f(\vec{r}_0) = \vec{u} \cdot \vec{\nabla} f(\vec{r}_0)$$

$$= |\vec{u}|\,|\vec{\nabla} f(\vec{r}_0)| \cos\theta$$

$$\| \quad C_{\vec{u}}(\vec{\nabla} f)$$

$$= |\vec{\nabla} f(\vec{r}_0)| \cos\theta$$

$D_{\vec{u}} f(\vec{r}_0)$ is greatest when $\theta = 0$



$$\vec{u} = \frac{\vec{\nabla} f(\vec{r}_0)}{|\vec{\nabla} f(\vec{r}_0)|} \quad (\text{if } \vec{\nabla} f(\vec{r}_0) \neq \vec{0})$$

$\vec{\nabla} f$ points in direction that f increases the fastest and $D_{\vec{u}} f = |\vec{\nabla} f(\vec{r}_0)|$

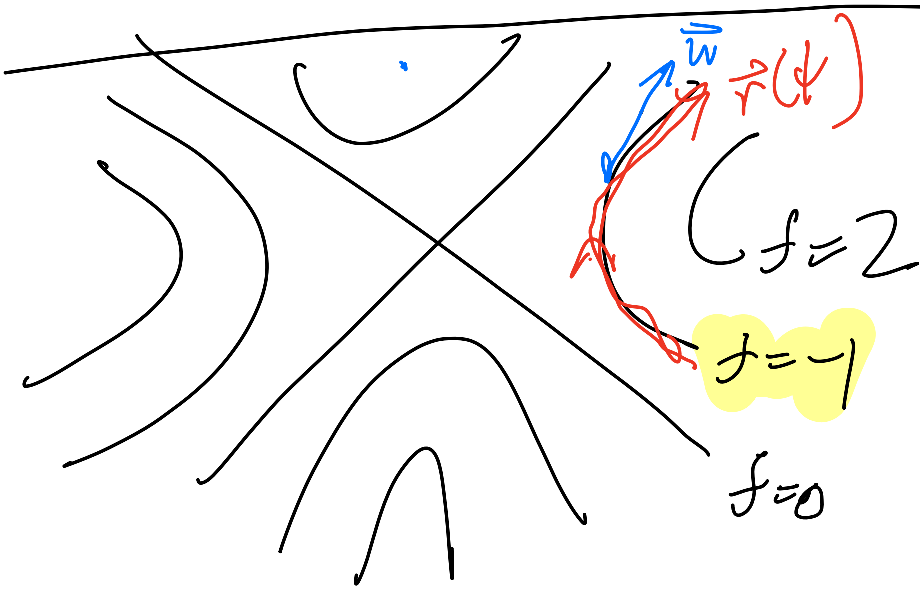
If $\vec{u} = -\frac{\vec{\nabla} f}{|\vec{\nabla} f|}$, $D_{\vec{u}} f = -|\vec{\nabla} f|$

If $\vec{u} \cdot \vec{\nabla} f = 0$, then $D_{\vec{u}} f = 0$

If $|\vec{u}| = 1$,

$$-|\vec{\nabla} f(\vec{r}_0)| \leq D_{\vec{u}} f(\vec{r}_0) \leq |\vec{\nabla} f(\vec{r}_0)|$$

$$D_{\vec{u}} f(\vec{r}_0) = 0 \iff \vec{u} \perp \vec{\nabla} f(\vec{r}_0)$$



Here, each contour is a curve on which f is constant

So if $\vec{u}$ tangent to contour,

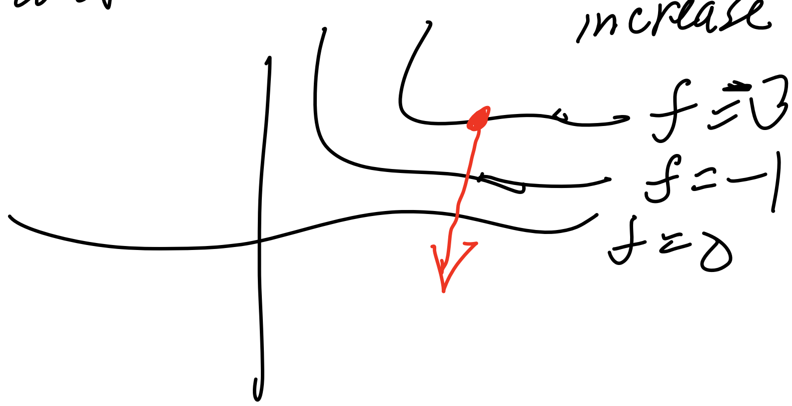
$$D_{\vec{u}} f = \left.\frac{d}{dt}\right|_{t=0} f(\vec{r}(t)) = 0$$

where $\vec{r}(t)$ is a parameterization of contour

$\Rightarrow \quad \vec{u} \cdot \vec{\nabla} f(\vec{r}_0) = 0$

if $\vec{u}$ is tangent to contour

$\Rightarrow \quad \vec{\nabla} f(\vec{r}_0)$ is normal to contour and in direction of maximal increase



$f(x,y,z) = x^2 + y^2 - z^2$

$f(1,1,1) = 1$

Contour of f containing $(1,1,1)$

is $x^2+y^2-z^2=1$

which is a surface "contour surface",

If $\vec{u}$ is tangent to surface,

then $D_{\vec{u}} f(1,1,1) = 0$

If $\vec{u} = \dfrac{\vec{\nabla} f(1,1,1)}{|\vec{\nabla} f(1,1,1)|}$, it is normal to contour

$\vec{n} = \pm \dfrac{\vec{\nabla} f}{|\vec{\nabla} f|}$

Here $\vec{\nabla} f = \langle 2x, 2y, -2z \rangle$

$\vec{\nabla} f(1,1,1) = \langle 2, 2, -2 \rangle$

$$\vec{n} = \frac{\vec{\nabla} f(1,1,1)}{|\vec{\nabla} f(1,1,1)|} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right\rangle$$

$\vec{\nabla} f(1,1,1)$ is normal to $f=1$ at $(1,1,1)$

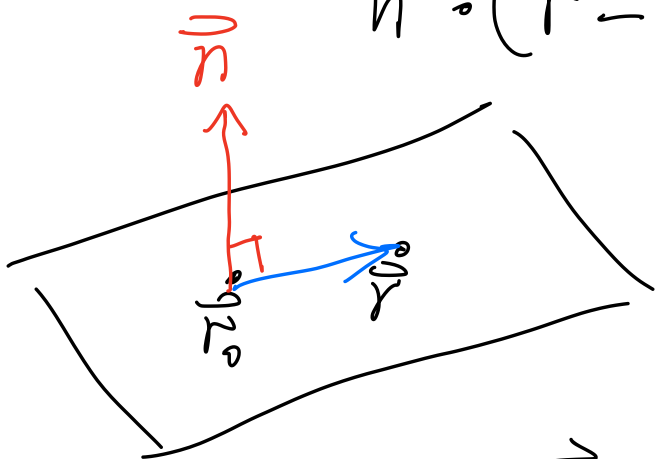
Equation of tangent plane to $f=1$ at $(1,1,1)$

Tangent plane at $(1,1,1)$

1) Contains $(1,1,1)$
2) $\vec{\nabla} f(1,1,1) = \langle 2, 2, -2 \rangle$ is normal to tangent plane

Recall that if $(x_0, y_0, z_0) = \vec{r}_0$ lies in a plane and $\vec{n}$ is normal to plane, then equation of plane is

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$



Here, $\vec{n} = \vec{\nabla} f(1,1,1) = \langle 2, 2, -2 \rangle$

Use $\vec{n} = \langle 1, 1, -1 \rangle$ instead

$\vec{r}_0 = \langle 1, 1, 1 \rangle$

$\vec{r} = \langle x, y, z \rangle$

$$0 = \vec{n} \cdot (\vec{r} - \vec{r}_0) = \langle 1, 1, -1 \rangle \cdot \langle x-1, y-1, z-1 \rangle$$

$$= x - 1 + y - 1 - z + 1 = x + y - z - 1$$

So tangent plane to contour $f=1$
where $f=x^2+y^2-z^2$, at $(1,1,1)$

is $\quad x+y-z-1=0$

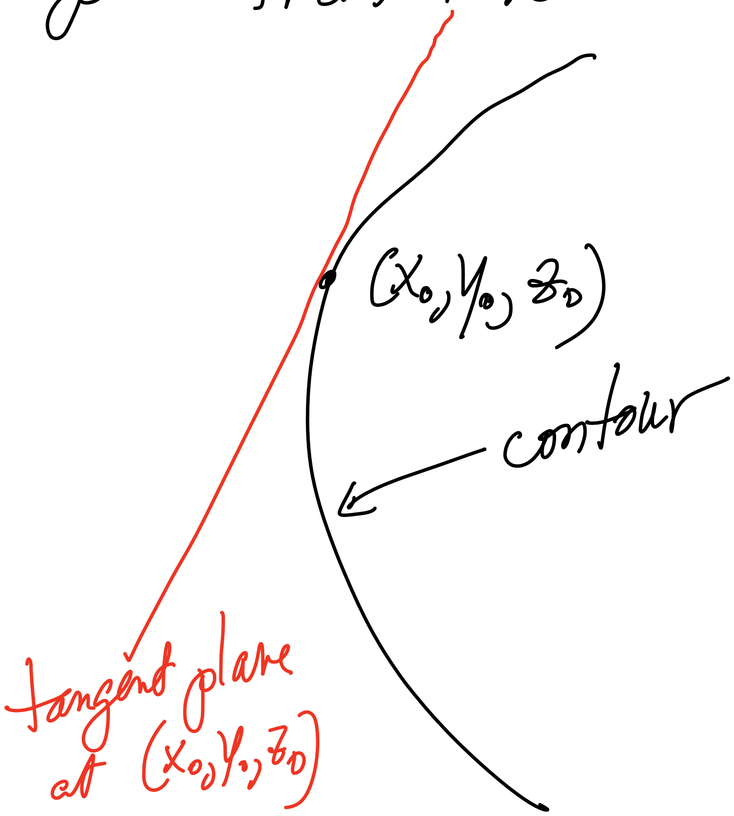
$1+1-1-1=0$ so $(1,1,1)$ lies in plane

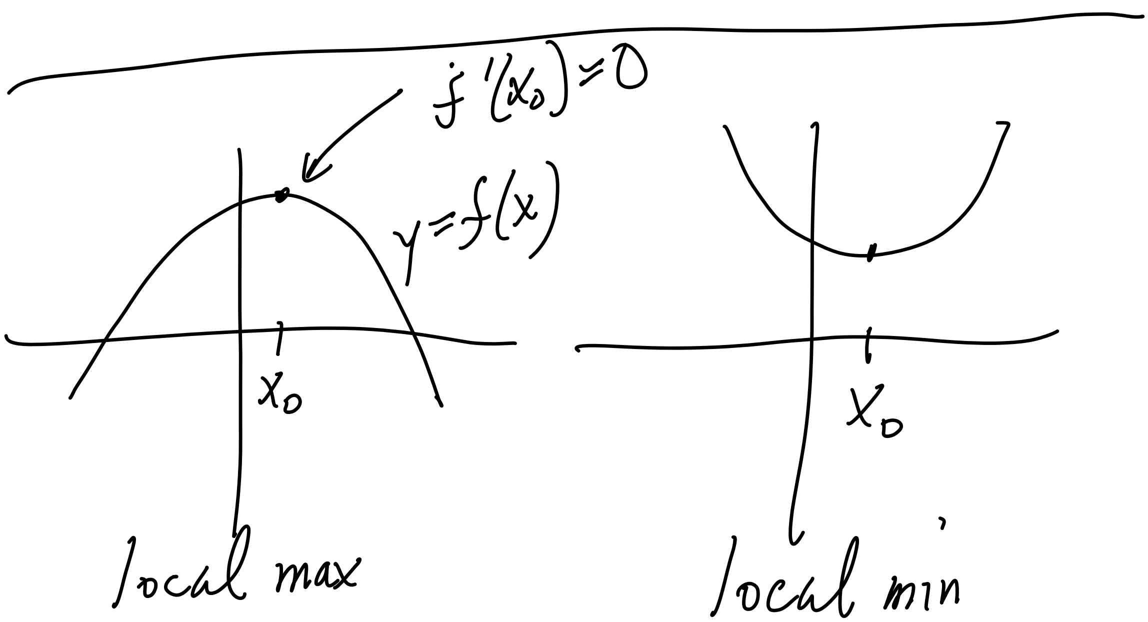
Coefficients are $\langle 1,1,-1\rangle$

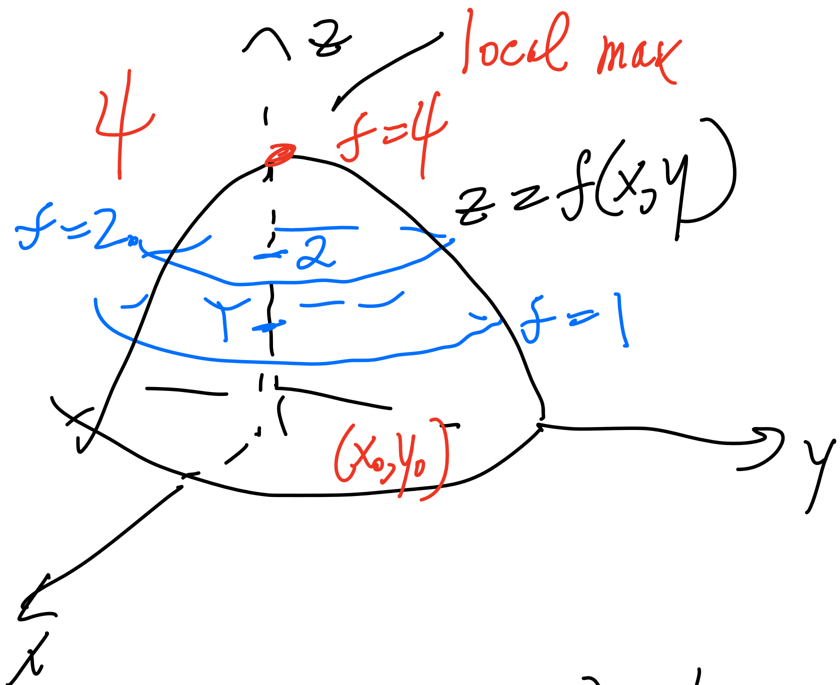
Given $f(x,y,z)$,
a contour $f(x,y,z)=c$,
and a point (x_0,y_0,z_0) on contour,
then $\vec{n}=\vec{\nabla}f(x_0,y_0,z_0)$
is normal to contour at (x_0,y_0,z_0)
and therefore normal to tangent plane to the contour at (x_0,y_0,z_0)

$\Rightarrow$ tangent plane has normal vector $\vec{\nabla}f\langle x_0,y_0,z_0\rangle$
and contains (x_0,y_0,z_0)

$\Longrightarrow$ can write equation for tangent plane from this

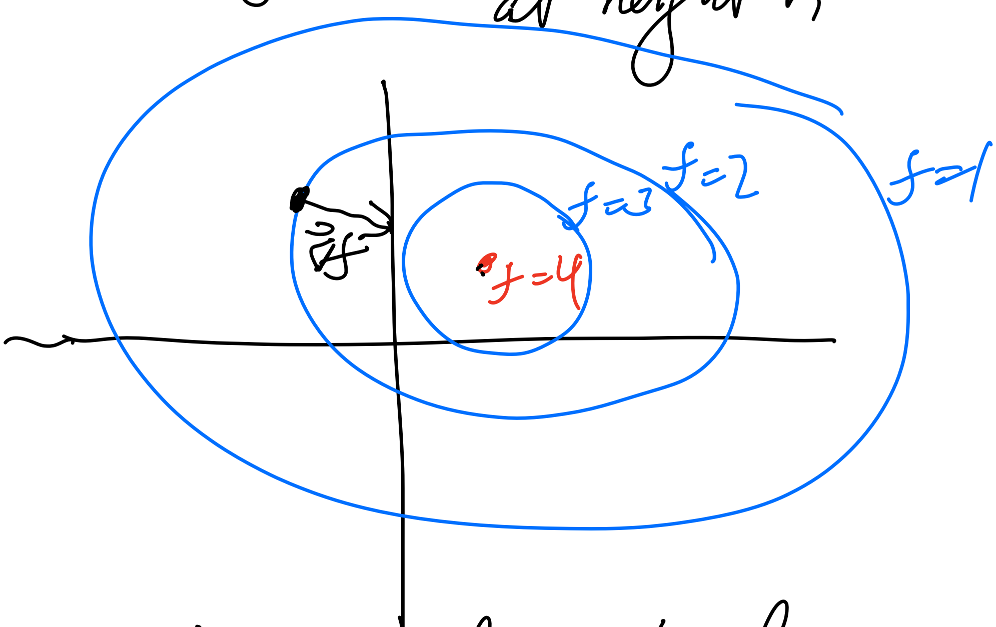




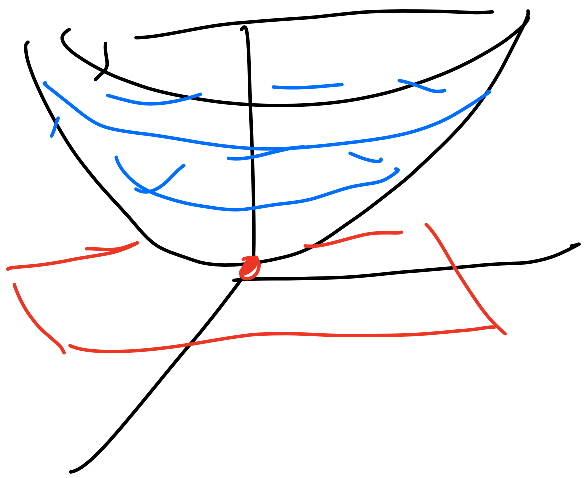


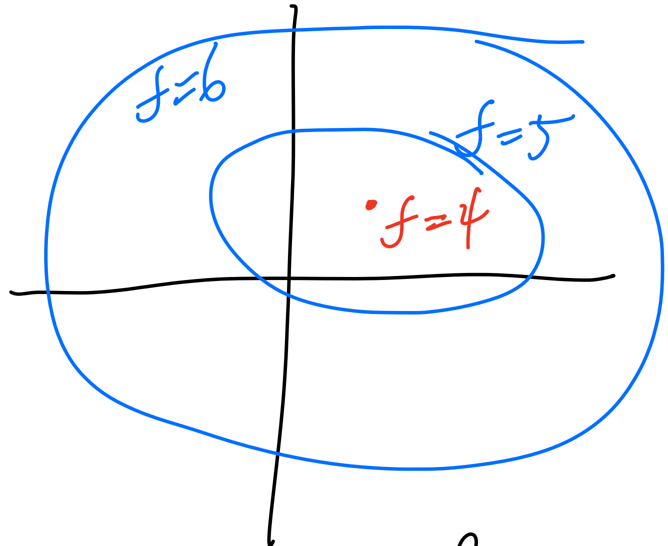
Level sets: $f(x,y) = h$

Slice graph with horizontal plane at height h



Contour map of a local max





Contour map of local minimum

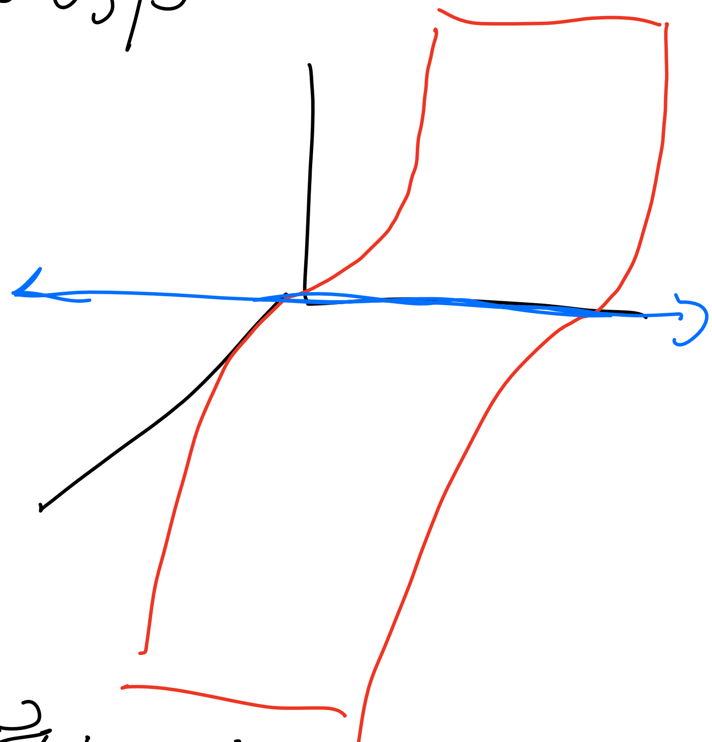
local max or min point has zero gradient

If f has a local max or min at (x_0, y_0), then $\vec{\nabla} f(x_0, y_0) = \vec{0}$

But converse is not necessarily true

If $\vec{\nabla} f(x_0, y_0) = \vec{0}$, (x_0, y_0) could be a local max or min but it could be something else

$f(x,y) = x^3$

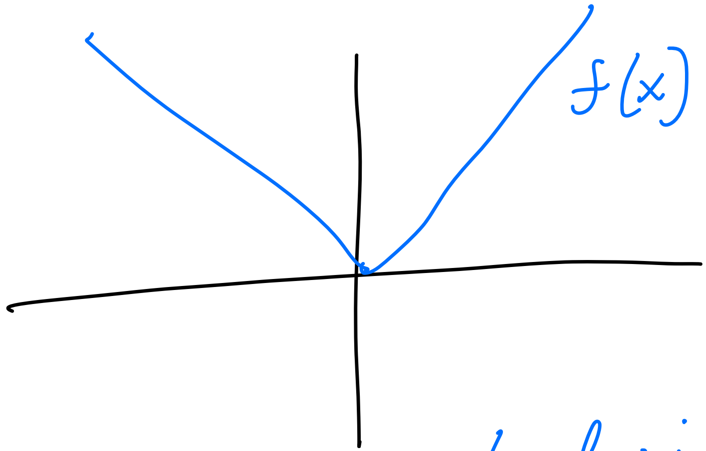


$\vec{\nabla} f(0,y) = \vec{0}$

$\vec{\nabla} f = \langle 3x^2, 0 \rangle$

$\vec{\nabla} f(0,y) = \langle 0, 0 \rangle$

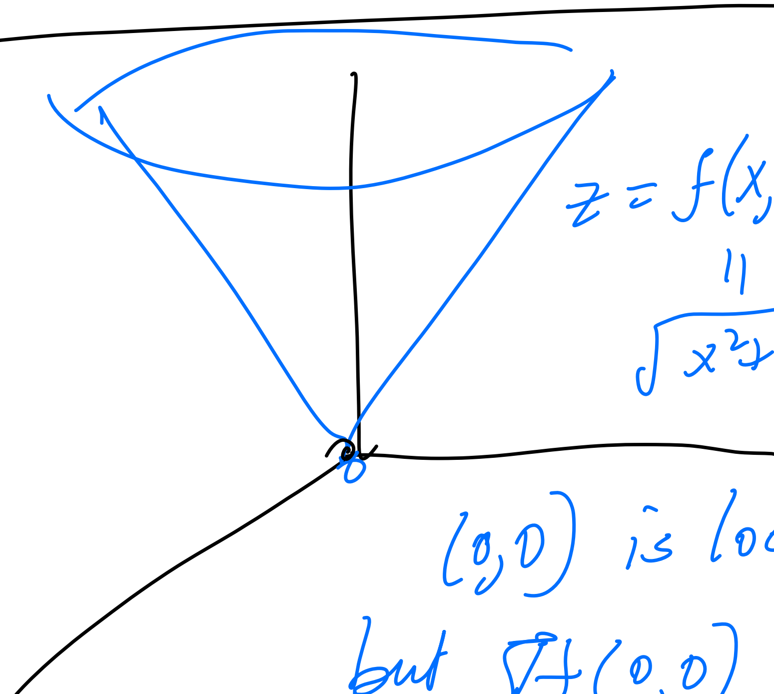
but no local max or min



$f(x) = |x|$

$x=0$ is a local min

but $f'(0)$ is undefined



$z = f(x,y) = \sqrt{x^2+y^2}$

$(0,0)$ is local min

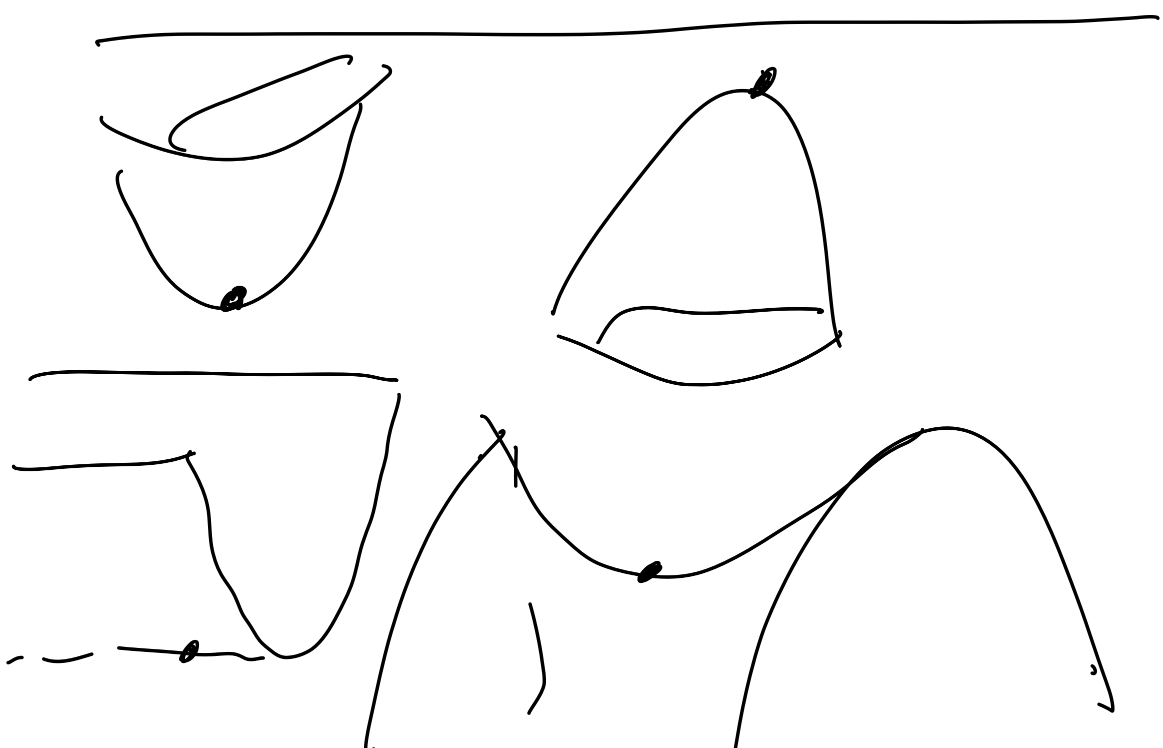
but $\nabla f(0,0)$ is undefined

Critical point of $f(x,y)$

(x_0,y_0) is a critical point of f if either $\vec{\nabla} f(x_0,y_0) = \langle 0,0\rangle$

or: $f_x(x_0,y_0)$ or $f_y(x_0,y_0)$ is undefined

(i.e., $\vec{\nabla} f(x_0,y_0)$ is undefined)



Many different kinds of critical points

Example $f(x,y) = x^2 + xy + y^2$

$\vec{\nabla} f(x,y) = \langle 2x+y,\ x+2y \rangle$

$\vec{\nabla} f(0,0) = \langle 0,0 \rangle$

Is it a local max, local min, saddle point, or something else?

2^{nd} derivative test

Based quadric surfaces

$z = x^2 + y^2$ local min

$z = -x^2 - y^2$ local max

$z = x^2 - y^2$ or $-x^2 + y^2$ saddle point

2nd derivatives of $f(x,y)$

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

"symmetric 2-by-2 matrix"

$$\det H = f_{xx} f_{yy} - (f_{xy})^2$$

If $\det H > 0$, then (x_0, y_0) is local max or min

If $\det H < 0$, then saddle point

If $\det H = 0$, inconclusive
