Partial derivatives

$$\underbrace{\begin{Bmatrix} C \\ B \\ A \end{Bmatrix}}_{\text{Inputs}} \begin{matrix} \to \\ \to \\ \to \end{matrix} \boxed{\;Q\;} \longrightarrow \underbrace{Q(C,B,A)}_{\text{Output}}$$

$$\frac{\partial Q}{\partial B} = Q_B \simeq \frac{\text{resulting change in } Q}{\text{small change in } B}$$

with A, C held constant

$$\underset{\text{due to change in } B \text{ only}}{\text{change in } Q} \simeq (Q_B)(\text{change in } B)$$

$$\text{change in } Q \simeq (Q_A)(\text{change in } A) + (Q_B)(\text{change in } B) + (Q_C)(\text{change in } C)$$

linear function of changes in A, B, C

Linear Approximation

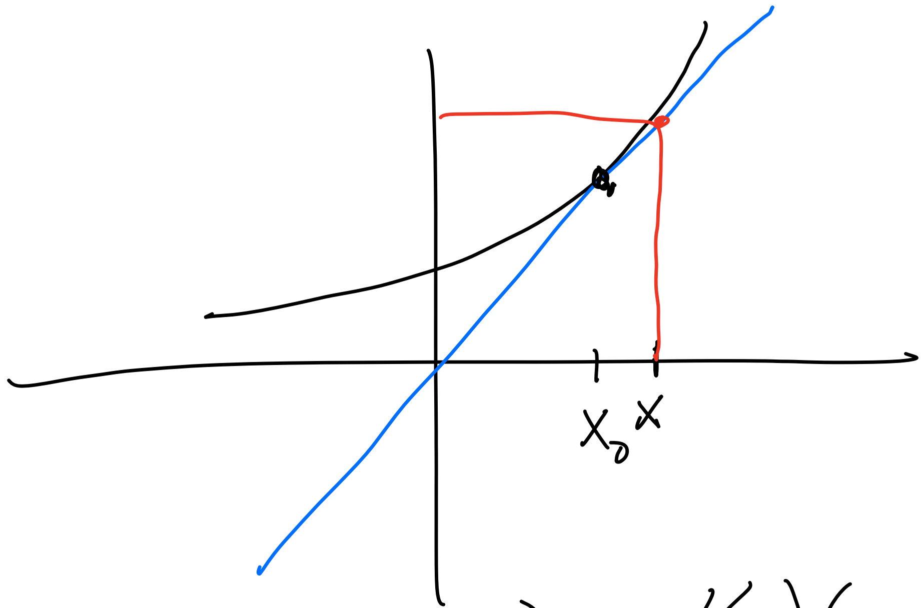
Example: $f(x,y)$ function

$f(1,-1) = 2$, $f_x(1,-1) = -3$,
$f_y(1,-1) = 1$

$$\text{Change in } f \simeq (f_x)(\text{small change in } x)$$
$$(f_y)(\text{small change in } y)$$

$$f(1.1,-1.2) - f(1,-1)$$
$$\simeq f_x(1,-1)(1.1-1)$$
$$+ f_y(1,-1)(-1.2-(-1))$$

$$f(1.1,-1.2) \simeq 2 + (-3)(0.1)$$
$$+ 1(-0.2)$$
$$= 2 - 0.3 - 0.2 = 1.5$$



$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

For $f(x,y)$

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

Here x_0, y_0 are held fixed (i.e. constants)

$\Rightarrow f(x_0,y_0), f_x(x_0,y_0), f_y(x_0,y_0)$ are also constants

RHS is a linear function of x, y

Back to example

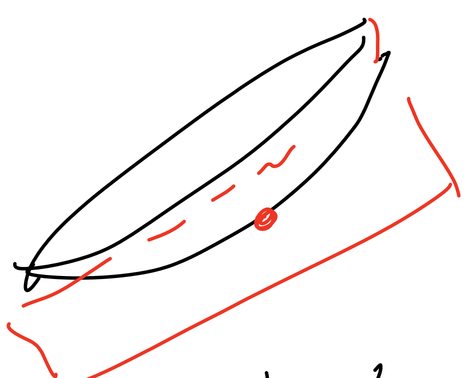
$$f(x,y) \cong f(1,-1) + f_x(1,-1)(x-1) + f_y(1,-1)(y-(-1))$$

$$= 2 - 3(x-1) + 1(y+1)$$

$$f(x,y) \cong -3x + y + 4$$

if (x,y) is close to $(1,-1)$

Graph of f $z = f(x,y)$ $\cong$ Graph of RHS $\underbrace{z = -3x + y + 4}_{\text{plane}}$



Two graphs touch at $(1,-1)$

Plane is tangent to graph of f at $(1,-1)$

Given $f(x,y)$ and a point (x_0,y_0) in the domain of f, f_x, f_y

Tangent plane to graph of f at (x_0,y_0) is

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$= ax + by + c$$

$a = f_x(x_0,y_0)$, $b = f_y(x_0,y_0)$

$$c = f(x_0,y_0) - f_x(x_0,y_0)x_0 - f_y(x_0,y_0)y_0$$

Equation of tangent plane at (x_0,y_0) is an approximation of near (x_0,y_0)

Called linear approximation

Differential of a function

Calculus 2

$$\int_1^2 (1+x^2)^3 x\,dx \qquad \boxed{u} = 1+x^2$$

function of x

$$\boxed{du} = 2x\,dx$$

differential of $u(x)$

Differential of $f(x,y,z)$

$$df = f_x\,dx + f_y\,dy + f_z\,dz$$

$$f(x,y,z) = xy^2z^3$$

$$df = d(xy^2z^3)$$

$$= y^2z^3\,dx + xz^3(2y\,dy) + xy^2(3z^2\,dz)$$

$$= y^2z^3\,dx + 2xyz^3\,dy + 3xy^2z^2\,dz$$

This is also a formula for linear approximation

Estimate $f(1.1, 1.9, 2.1)$

$$\approx$$

$$f(1, 2, 2)$$

$$1(2^{1/2})(2^3) = 32$$

df = change in f

$$df = 4(8)(\text{change in } x) + 2(1)(2)(8)(\text{change in } y) + 3(1)(4)(4)(\text{change in } z)$$

$$= 32(0.1) + 32(-0.1) + 48(0.1)$$

$$= 3.2 - 3.2 + 4.8$$

$$f(1.1, 1.9, 2.1) \approx f(1, 2, 2) + \text{change in } f$$

$$= 32 + 4.8 = 36.8$$

$$d(f+g) = df + dg$$
$$d(fg) = g\,df + f\,dg$$
$$d\left(\frac{f}{g}\right) = \frac{g\,df - f\,dg}{g^2}$$

Implicit differentiation

$z = f(x,y)$ satisfies

$$x^2 + y^2 + z^2 = 3xyz$$

Find f_x, f_y:

Compute differential of f and of equation

$$dz = df = f_x\,dx + f_y\,dy$$

$$2x\,dx + 2y\,dy + 2z\,dz$$
$$= 3yz\,dx + 3xz\,dy + 3xy\,dz$$

Collect like terms

$$(2x-3yz)\,dx + (2y-3xz)\,dy = (3xy-2z)\,dz$$

$$\underbrace{dz}_{df} = \underbrace{\frac{2x-3yz}{3xy-2z}}_{f_x}\,dx + \underbrace{\frac{2y-3xz}{3xy-2z}}_{f_y}\,dy$$

Works only if you know not only the values of x, y but also the value of z

Example

$(1,1,1)$ satisfies equation

$$f_x(1,1) = \frac{2-3}{3-2} = -1$$

$$f_y(1,1) = \frac{2-3}{3-2} = -1$$

If (x,y,z) satisfies $3xy-2z=0$,

(and equation defining f)

then f_x, f_y are undefined

for those values

of x and y.

Estimate $f(0.9, 1.2)$

$$dz = -dx - dy$$

$$f(0.9,1.2) - f(1,1) \approx -(0.9-1) - (1.2-1)$$

$$f(0.9,1.2) \approx 1 + 0.1 - 0.2 = 0.9$$

Ideal Gas Law

$$\frac{PV}{T} = k$$

P = pressure

V = volume

T = temperature

$$P = \frac{kT}{V}, \quad V = \frac{kT}{P}, \quad T = \frac{PV}{k}$$

$$P_V, P_T, T_V, T_P, V_T, V_P$$

$$PV = kT$$

$$V\,dP + P\,dV = k\,dT$$

Suppose $V(P, T)$

$$P\,dV = k\,dT - V\,dP$$

$$dV = \frac{h}{P}\,dT - \frac{V}{P}\,dP$$

$$V_T = \frac{h}{P}, \quad V_P = -\frac{V}{P}$$

Suppose $T(V,P)$

$$dT = \frac{V}{h}\,dP + \frac{P}{h}\,dV$$

$$T_P = \frac{V}{h}, \quad T_V = \frac{P}{h}$$

Chain Rule

Consider $f(\cdot,\cdot)$, $g(\cdot,\cdot)$, $h(\cdot,\cdot)$

$$P(x,y) = h\big(f(x,y), g(x,y)\big)$$

$$P_x = h_u\big(f(x,y),g(x,y)\big)f_x(x,y) + h_v\big(f(x,y),g(x,y)\big)g_x(x,y)$$

Write $h(u,v)$, $u = f(x,y)$, $v = g(x,y)$

Start with $f(x,y)$

$$Q(x) = f(x, 1-x)$$

$$Q'(x) = ?$$

$$Q'(2) = ?$$

$$f(x,y) = xe^{-y}$$

$$Q(p) = f(p, 1-p)$$

$$dQ = Q'(p)\,dp$$

$$Q(p) = f(x,y)$$

where $\boxed{x = p,\ y = 1-p}$

Compute differentials

$$dQ = df$$

$$Q'(p)\,dp = f_x\,dx + f_y\,dy$$

$$dx = dp$$
$$dy = -dp$$

$$Q'(p)\,dp = f_x(x,y)\,dx + f_y(x,y)\,dy$$
$$= f_x(p, 1-p)(dp)$$
$$+ f_y(p, 1-p)(-dp)$$
$$= \left[f_x(p, 1-p) - f_y(p, 1-p)\right] dp$$

$$\Longrightarrow \quad Q'(p) = f_x(p, 1-p) - f_y(p, 1-p)$$

$$f(x,y) = (2xy + 3x)^2 e^{x^2 - y^2}$$

$f_x, f_y = ?$

Outer

$$\boxed{f(x,y) = p^2 e^q}$$

Inside functions

where $\boxed{p = 2xy + 3x, \quad q = x^2 - y^2}$

$$df = f_x\,dx + f_y\,dy$$

$$= 2pe^q\,dp + p^2e^q\,dq$$

$$dp = 2y\,dx + 2x\,dy + 3dx$$

$$= (2y+3)\,dx + 2x\,dy$$

$$dq = 2x\,dx - 2y\,dy$$

Plug formulas for p, q, dp, dq into df

$$df = 2(2xy+3x)e^{x^2-y^2}\left((2y+3)dx+2x\,dy\right)$$

$$+ (2xy+3x)^2e^{x^2-y^2}\left(2x\,dx-2y\,dy\right)$$

$$= [\text{ formula in } x,y \text{ }]\,dx$$

$$[\text{ formula in } xy \text{ }]\,dy$$

$$= f_x(x,y)\,dx + f_y(x,y)\,dy$$

More examples in slides
(Try to do computations yourself and then compare to slides)

Outside function $f(p,q)$

Inside functions

$$p = p(x,y) \qquad p = x+y$$
$$q = q(x,y) \qquad q = x-y$$

example

Composition

$$Q(x,y) = f(p(x,y), q(x,y))$$

Find Q_x, Q_y

Differential of outside

$$df = f_p\,dp + f_q\,dq$$

Differentials of inside

$$dp = p_x\,dx + p_y\,dy$$
$$dq = q_x\,dx + q_y\,dy$$

$$df = f_p(p,q)\,dp + f_q(p,q)\,dq$$
$$= f_p\big(p(x,y),q(x,y)\big)\big(p_x(x,y)\,dx + p_y(x,y)\,dy\big)$$
$$+ f_q\big(p(x,y),q(x,y)\big)\big(q_x(x,y)\,dx + q_y(x,y)\,dy\big)$$
$$= \left(f_p\,p_x + f_q\,q_x\right)dx$$
$$+ \left(f_p\,p_y + f_q\,q_y\right)dy$$
$$= f_x\,dx + f_y\,dy = Q_x\,dx + Q_y\,dy$$

$$Q(x,y) = f(p,q)$$
where $p = p(x,y)$, $q = q(x,y)$

$$\Rightarrow \left.\begin{aligned} Q_x &= f_p p_x + f_q q_x \\ Q_y &= f_p p_y + f_q q_y \end{aligned}\right\}$$ Derivatives of outside times Derivatives of inside

---

$$h(p,q) = (p+q)e^{p-q}$$

$$\boxed{\begin{aligned} h_p &= (1)e^{p-q} + (p+q)e^{p-q} \\ &= (1+p+q)e^{p-q} \end{aligned}}$$

$$\boxed{\begin{aligned} h_q &= (1)e^{p-q} + (p+q)\left(-e^{p-q}\right) \\ &= (1-p-q)e^{p-q} \end{aligned}}$$

$$f(x,y) = xe^y$$

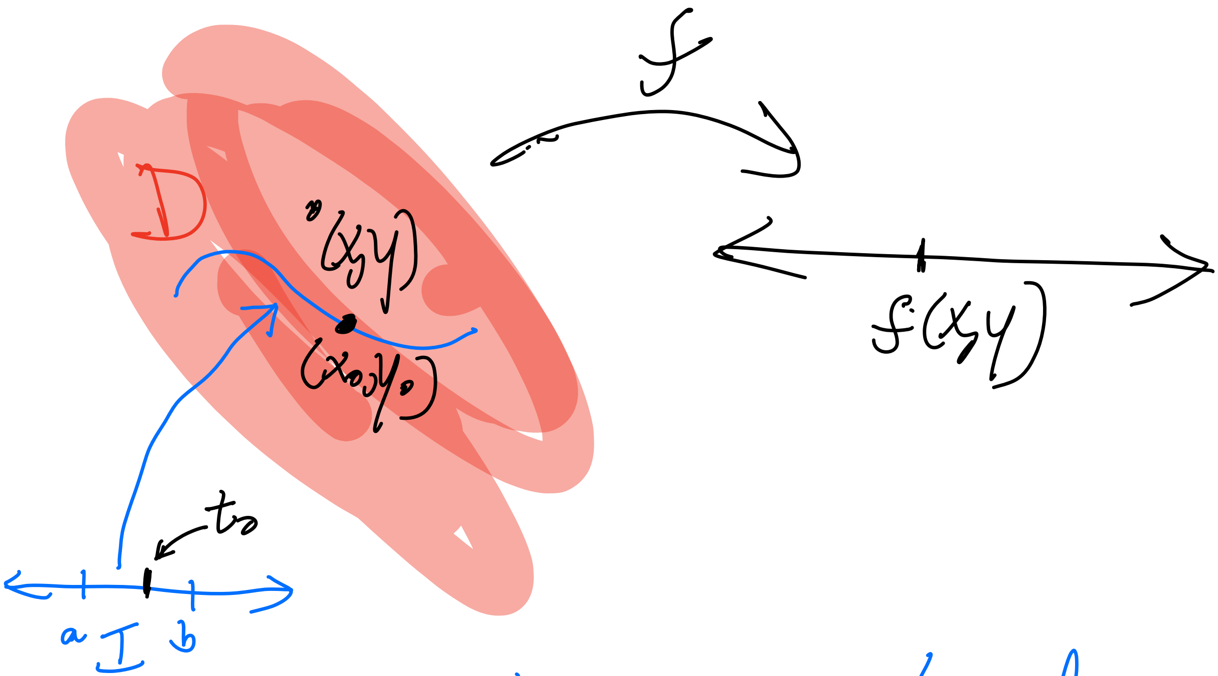
$$x = p+q, \quad y = p-q$$

$$dx = dp + dq, \quad dy = dp - dq$$

$$dh = h_p\, dp + h_q\, dq$$

but $h(p,q) = xe^y$, where

$$\boxed{\begin{aligned} dh &= e^y dx + xe^y dy \\ &= e^{p-q}(dp+dq) + (p+q)e^{p-q}(dp-dq) \\ &= e^{p-q}\big((1+p+q)\,dp + (1-p-q)\,dq\big) \\ h_p &= e^{p-q}(1+p+q) \\ h_q &= e^{p-q}(1-p-q) \end{aligned}}$$



$\vec{r}(t)$, $a \le t \le b$ is parameterized curve

Focus on one point on the curve

$$\vec{r}(t_0) = (x_0, y_0)$$

Derivative of f along curve at (x_0, y_0) is

$$\frac{d}{dt}\Big(f(\vec{r}(t))\Big)\Big|_{t=t_0} = \frac{d}{dt}\Big(f(x(t), y(t))\Big)\Big|_{t=t_0}$$

$$\boxed{\left.\frac{d}{dt}\right|_{t=t_0} f(x(t),y(t))}$$

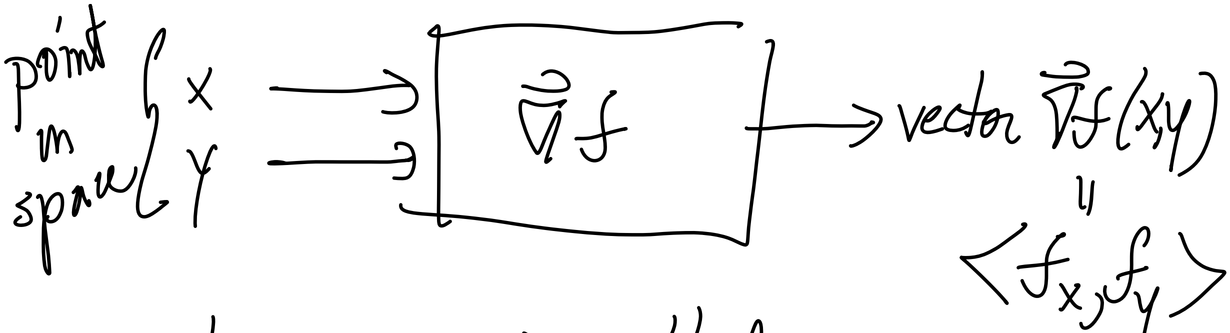
$$= \left.\left[f_x(x(t),y(t))\,x'(t) + f_y(x,y)\,y'(t)\right]\right|_{t=t_0}$$

$$= f_x(x_0,y_0)\,x'(t_0) + f_y(x_0,y_0)\,y'(t_0)$$

$$= \underbrace{\underbrace{\langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle}_{\substack{\text{does not depend curve,}\\ \text{only on } f}}}_{\text{depends on } f \text{ only}} \cdot \underbrace{\underbrace{\langle x'(t_0), y'(t_0)\rangle}_{\vec{r}\,'(t_0) \,=\, \vec{v}(t_0)}}_{\substack{\text{depends}\\ \text{on curve}\\ \text{only}}}$$

Define the gradient of $f(x,y)$ to be

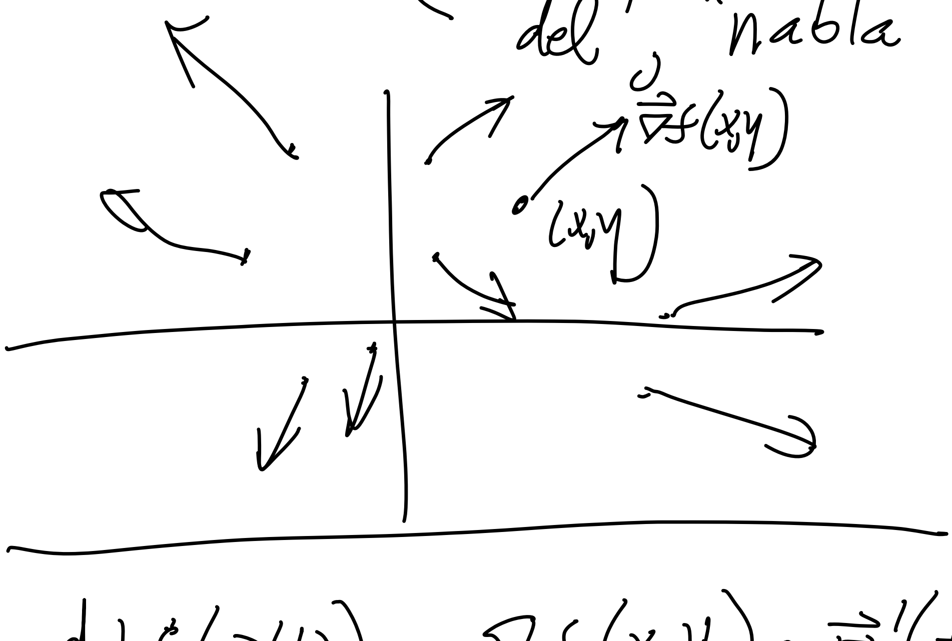
$$\vec{\nabla} f(x,y) = \langle f_x, f_y\rangle = \vec{i} f_x + \vec{j} f_y$$



Such a map is called a vector field

It assigns to each point (x,y) in the domain of f a vector called $\vec\nabla f(x,y)$

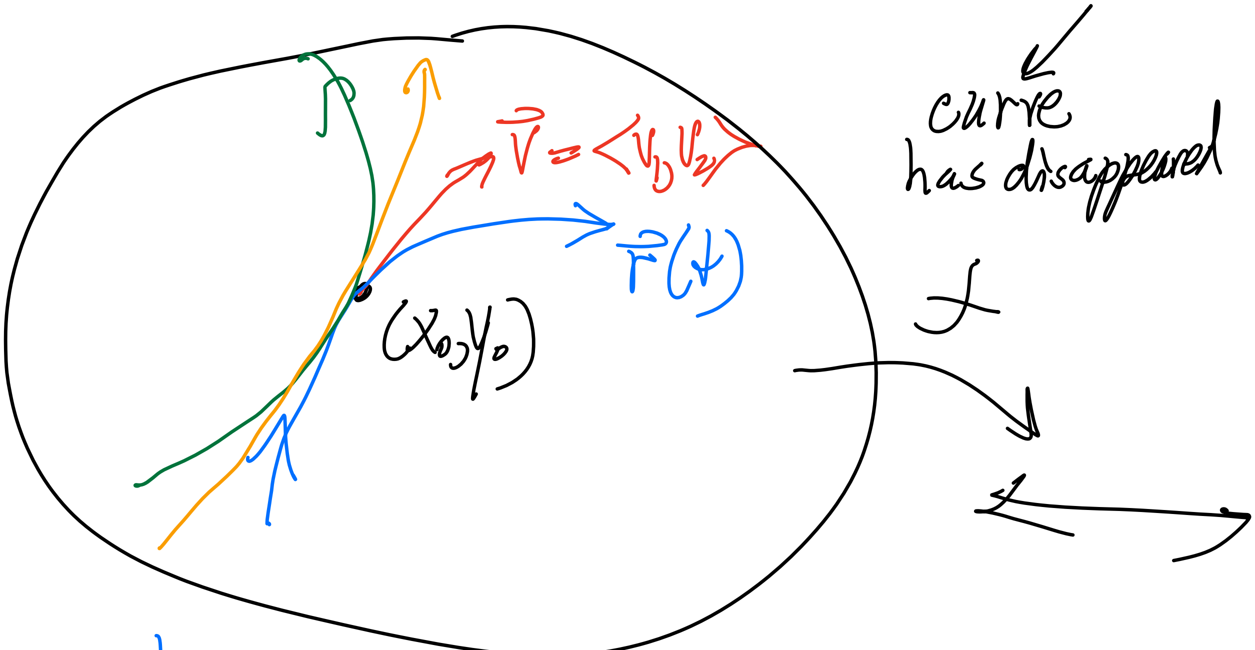
"del", "nabla"



$$\frac{d}{dt}\bigg|_{t=t_0} f(\vec r(t)) = \nabla f(x_0,y_0)\cdot \vec r\,'(t_0)$$

$$\vec{r}\,'(t_0) = \vec{v}(t_0) = \langle v_1, v_2 \rangle$$

$$\left.\frac{d}{dt}\right|_{t=t_0} f(\vec{r}(t)) = \vec{\nabla} f(x_0, y_0) \cdot \underbrace{\langle v_1, v_2 \rangle}_{\text{curve has disappeared}}$$

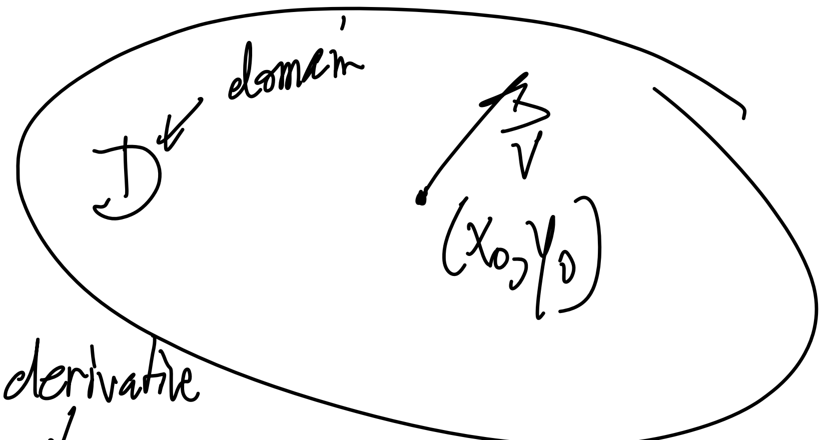


where $\vec{r}\,'(0) = \vec{v}$

$\vec{r}(0) = (x_0, y_0)$

$$\left.\frac{d}{dt}\right|_{t=0} f(\vec{r}(t)) = \vec{\nabla} f(x_0, y_0) \cdot \vec{v}$$

We can define the directional derivative of $f(x,y)$ at (x_0,y_0) in the direction $\vec{v}$



derivative

$$D_{\vec{v}} f(x_0,y_0) = \left.\frac{d}{dt}\right|_{t=0} f(\vec{r}(t))$$

where $\vec{r}(0) = (x_0,y_0)$, $\vec{r}\,'(0) = \vec{v}$

$$D_{\vec{v}} f(x_0,y_0) = \vec{\nabla} f(x_0,y_0) \cdot \vec{v}$$