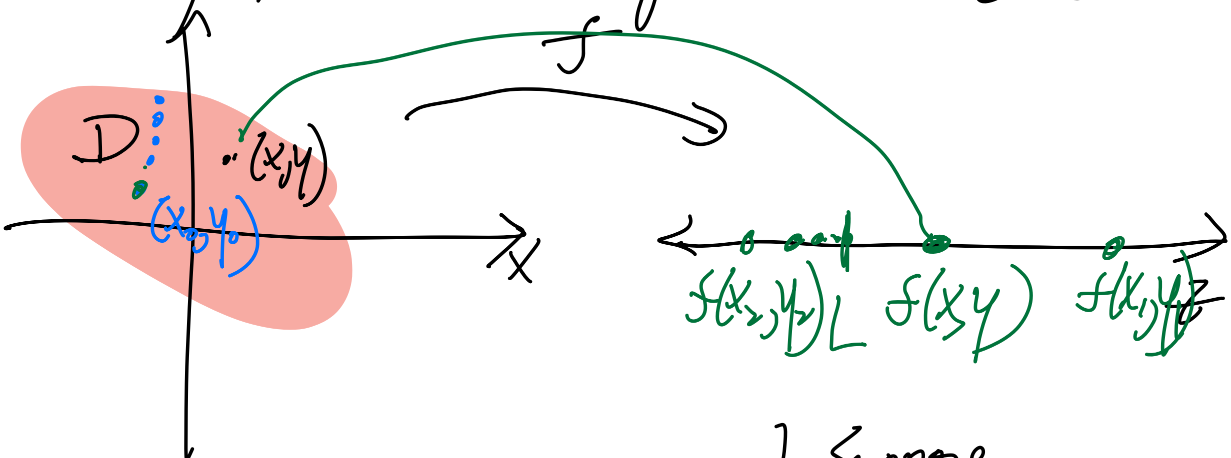
Lecture is Tuesday 10/12

Midterm Monday, October 25

Limit of function

Functions of 2 variables

Much more complicated 1 variable case




$(x_0,y_0)$ may or may be in domain of $f$

Let $(x_1,y_1),(x_2,y_2),\ldots$ be a sequence of points in domain of $f$

Suppose $(x_1,y_1),\ldots$

$\downarrow$

$(x_0,y_0)$

$$\lim_{k\to\infty}(x_k,y_k)=(x_0,y_0)$$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L$$

if $\lim_{k\to\infty} f(x_k, y_k) = L$

for every possible sequence $(x_k, y_k) \to (x_0, y_0)$

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## Informal rules on limits

Consider $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$

Possibities:

1) Equal to some number $L$
2) Undefined

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- Assume $f$ is given by a single formula

$$f(x,y) = e^{-x\sin y} + y^2\left(\frac{x+1}{y+1}\right)$$

$$f(x,y) = \begin{cases} x^2 y & \text{if } x>0 \text{ and } y<0 \\ \dfrac{x}{x^2-y^2} & \text{otherwise} \end{cases}$$

1) First, try calculating $f(x_0, y_0)$
If this works, then the limit is $f(x_0, y_0)$

$$\lim_{(x,y)\to(1,1)} \frac{x^2-y^2}{x^2+y^2} \stackrel{?}{=} \frac{1-1}{1+1}$$
$$= 0$$

$\Rightarrow$ limit is 0

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If $f(x_0, y_0) = \frac{\text{nonzero}}{0}$, $\sqrt{\text{negative number}}$
or $\ln(\text{negative number})$
then limit is undefined

$$1 = \frac{2}{2} = \frac{2}{0} \cdot \frac{0}{2}$$

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If $f(x_0, y_0) = \frac{0}{0}$ or $\frac{\infty}{\infty}$
then need to investigate further

## Example

$$\lim_{\substack{(x,y)\to(1,2)\\ \| \\ (x_0,y_0)}} \frac{1+x+y}{3-x-y} = \frac{1+1+2}{3-1-2} = \frac{4}{0}$$

$\Longrightarrow$ limit is undefined

$$\lim_{(x,y)\to(1,2)} \frac{x^2-y^2}{x^2+y^2} = \frac{1-4}{1+4} = \frac{-3}{5}$$

$\Longrightarrow$ limit exists and is equal to $\frac{-3}{5}$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \frac{0}{0}$$

Need to investigate

# Detecting undefined limits

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

Suppose there is a sequence

$$(x_n,y_n)\longrightarrow(x_0,y_0)$$

such that $f(x_n,y_n)$ is undefined for all $(x_n,y_n)$

$\Longrightarrow$ the limit is undefined

$$\lim_{(x,y)\to(1,1)} \frac{x^2+y^2}{x^2-y^2} =$$

Undefined if denominator is zero but numerator is not

$$\Rightarrow x^2-y^2=0=(x+y)(x-y)$$

$$\Longleftrightarrow y=\pm x$$

$$\left(1+\frac{1}{n}, 1+\frac{1}{n}\right)\longrightarrow(1,1)$$

but $\dfrac{x_n^2+y_n^2}{x_n^2-y_n^2} = \dfrac{(1+\frac{1}{n})^2+(1+\frac{1}{n})^2}{0}$ is undefined

So limit is undefined

$$\frac{1}{n} \to 0 \qquad \frac{1}{n^2} \to 0$$

$$x_n = 2 + \frac{1}{n} \to 2$$

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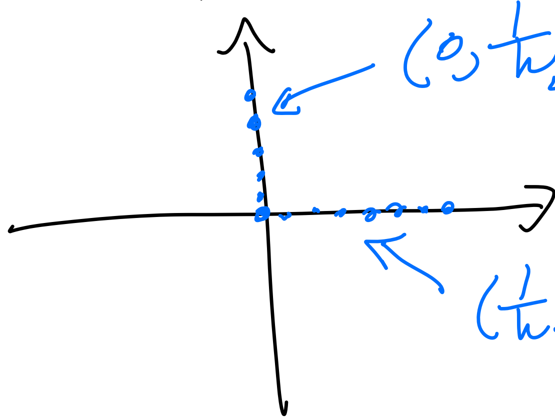
$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

$$\left.\left(0,\frac{1}{n}\right) \to (0,0)\right\} S1$$



$$\left.\left(\frac{1}{n},0\right) \to (0,0)\right\} S2$$

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$S1: \quad (x_n, y_n) = \left(0, \frac{1}{n}\right)$

$$\frac{x_n^2 - y_n^2}{x_n^2 + y_n^2} = \frac{0 - \frac{1}{n^2}}{0 + \frac{1}{n^2}} = -1 \to -1$$

$S2: (x_n, y_n) = (\frac{1}{n}, 0)$

$$\frac{x_n^2 - y_n^2}{x_n^2 + y_n^2} = \frac{\frac{1}{n^2} - 0}{\frac{1}{n^2} + 0} = 1$$

$S1 \longrightarrow -1$ and $S2 \longrightarrow +1$

$\Longrightarrow$ limit undefined

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$$\lim_{(x,y)\to(0,0)} \frac{x+2y}{x^2+y^2}$$

$\longrightarrow 0$ faster than

$$\| $$

$x = \frac{1}{n}, y = 0 \quad \frac{\frac{1}{n}}{\frac{1}{n^2}} = n \longrightarrow +\infty$

$\Longrightarrow$ limit ~~exis~~ undefined

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$$\left.\lim_{(x,y)\to 0} \frac{x^2+xy}{x^2+y^2}\right\}$$ degrees are equal

Try: $(x_n, y_n) = (\frac{1}{n}, 0)$ and $(0, \frac{1}{n})$

if different answers, then limit is undefined

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}:$$

$(x_n,y_n) = \left(\frac{1}{n},0\right)$, $(x_n,y_n) = \left(0,\frac{1}{n}\right)$

have limiting value of $0$

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$(x_n,y_n) = \left(\frac{a}{n},\frac{b}{n}\right)$ $a,b$ are unspecified constants

$$\frac{x_n y_n}{x_n^2+y_n^2} = \frac{\frac{a}{n}\cdot\frac{b}{n}}{\frac{a^2}{n^2}+\frac{b^2}{n^2}} = \frac{ab}{a^2+b^2}$$

$(a,b)=(1,1)$
$(a,b)=(3,4)$ give different limits

$\Longrightarrow$ limit is undefined

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$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+3y^2}$$

- Denominator $\neq 0$ if $(x,y)\neq(0,0)$
- $x^2+3y^2 \geqslant x^2+y^2$
- $|x|,|y| \leqslant \sqrt{x^2+y^2}$ $\Rightarrow$ $\sqrt{x^2}=|x|$, $\sqrt{y^2}=|y|$

$$\left|\frac{x^2y}{x^2+3y^2}\right| \le \frac{|x|^2|y|}{x^2+y^2} \le \frac{(x^2+y^2)(x^2+y^2)^{1/2}}{x^2+y^2}$$

$$= (x^2+y^2)^{1/2}$$

$$\left|\frac{x^2y}{x^2+3y^2}\right| \to 0 \qquad (x^2+y^2)^{1/2} \to 0$$

as $(x,y) \to 0$

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$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+3y^2} = 0$$

$$\lim_{(x,y)\to(x_0,y_0)} (\text{formula in } x \text{ and } y)$$

1) Plug $(x_0,y_0)$ into formula
   If it works, value is limit

2) If you get undefined,
   the limit is undefined

3) If you get $\frac{0}{0}$ or $\frac{\infty}{\infty}$, go
   to next steps

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$$\lim_{(x,y)\to(0,0)} \frac{\text{polynomial } \cancel{\text{of degree } p}}{\text{polynomial } \cancel{\text{of degree } q}}$$

1) if it's possible for numerator $\neq 0$
   but denominator $= 0$
   for a given $(x,y)$

$$\frac{x^2+y^2}{x^2-y^2}$$

then limit does not exist

Try to find $(x_k, y_k) \longrightarrow 0$
but denominator is always 0

2) if $p < q$, then usually
limit does not exist

Try $(x_k, y_k) = \left(\frac{a}{k}, \frac{b}{k}\right)$

3) If $p = q$, try $(x_k, y_k) = \left(\frac{a}{k}, \frac{b}{k}\right)$
Usually limit does not exist
because limit depends on a and b

4) If $p > q$ and denominator
is never zero

Use following identities to simplify limit
$\Downarrow$
replace by
something bigger

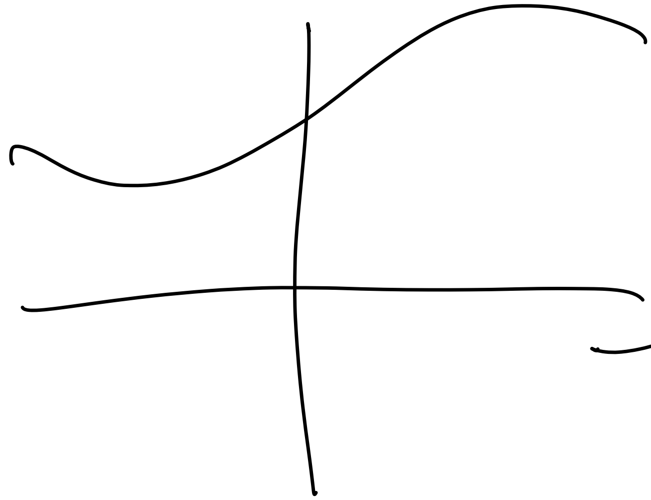
$$\left.\begin{aligned} |A+B| &\le |A| + |B| \\ |AB| &= |A|\,|B| \\ |A|, |B| &\le \sqrt{A^2+B^2} \end{aligned}\right\}$$

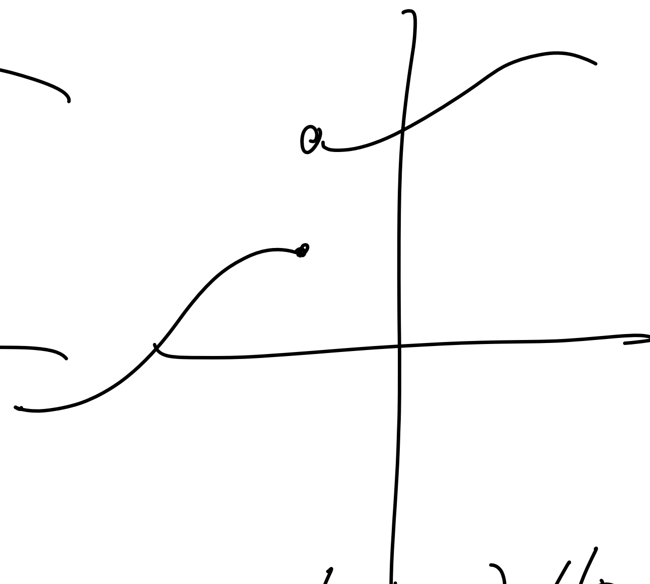
Use this to
replace limit by
a formula that
has only $(x^2+y^2)^{1/2}$ in it

In the above:

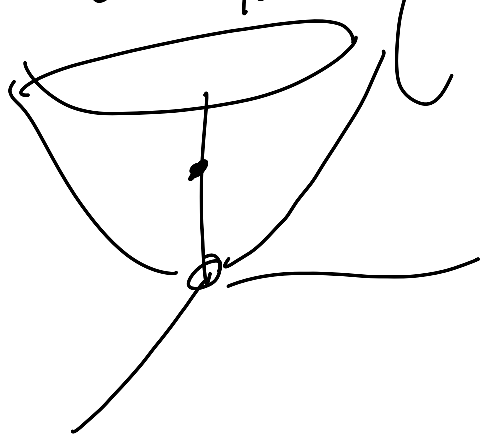
$p$ = degree of lowest order term in numerator

$q$ = degree of lowest order term in denominator





$$f(x,y) = \begin{cases} x^2+y^2 & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$



# Derivative of a function

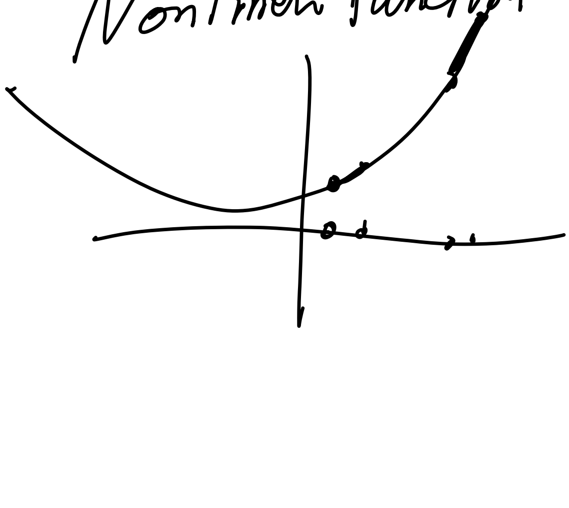
$$\text{input} \longrightarrow \boxed{f} \longrightarrow \text{output}$$

Derivative of $f \cong$ sensitivity of output to a small change in input

$$\cong \frac{\text{change in output}}{\text{change in input}}$$

Linear function : sensitivity same for every input

Nonlinear function : sensitivity depends starting input



# Linear approximation

$f(1) = 3, \; f'(1) = -2$

Estimate $f(0.8)$

change in input $= 0.8 - 1 = -0.2$

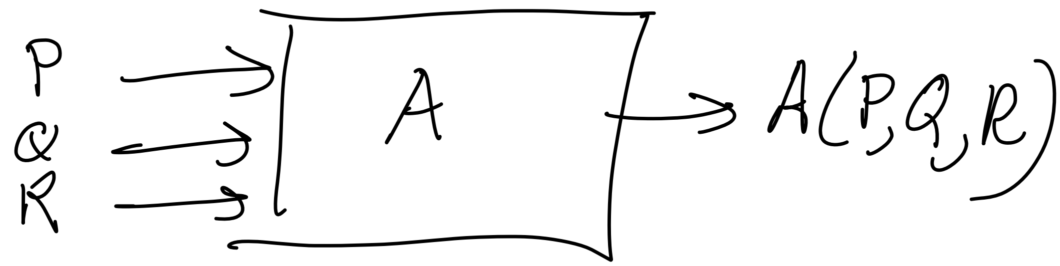
change in output $= f(0.8) - f(1)$

sensitivity $= f'(1) = -2$

$$-2 = \frac{f(0.8) - f(1)}{-0.2} = \frac{f(0.8) - 3}{-0.2}$$

$f(0.8) = 3 + 0.4 = 3.4$

# Functions with more than one input




## Partial derivative

$$\frac{\partial A}{\partial P}(P_0, Q_0, R_0) \approx \frac{\text{change in } A}{\text{small change in } P}$$

(with $Q$, $R$ unchanged)

Same for $\frac{\partial A}{\partial Q}$, $\frac{\partial A}{\partial R}$

Here, use $\partial$ instead of $d$

$$\frac{\partial A}{\partial P} \qquad\qquad \frac{dy}{dx}$$

Notation : $\frac{\partial A}{\partial P} = \partial_P A = A_P$

## Calculating partial derivatives

Ex : $A(P,Q,R) = PQ^2R^3$

$A_P$ = derivative of $A$ with respect to $P$ with $Q, R$ held constant

$A_P = Q^2R^3$

$A_Q = 2PR^3Q$

$A_R = 3PQ^2R^2$

$(A_P)_R = 3Q^2R^2$

$((A_P)_R)_R = 6Q^2R$

$(A_R)_P = 3Q^2R^2$

Equal

Mixed partials commute

$$(A_P)_Q = (A_Q)_P = A_{PQ} = A_{QP}$$

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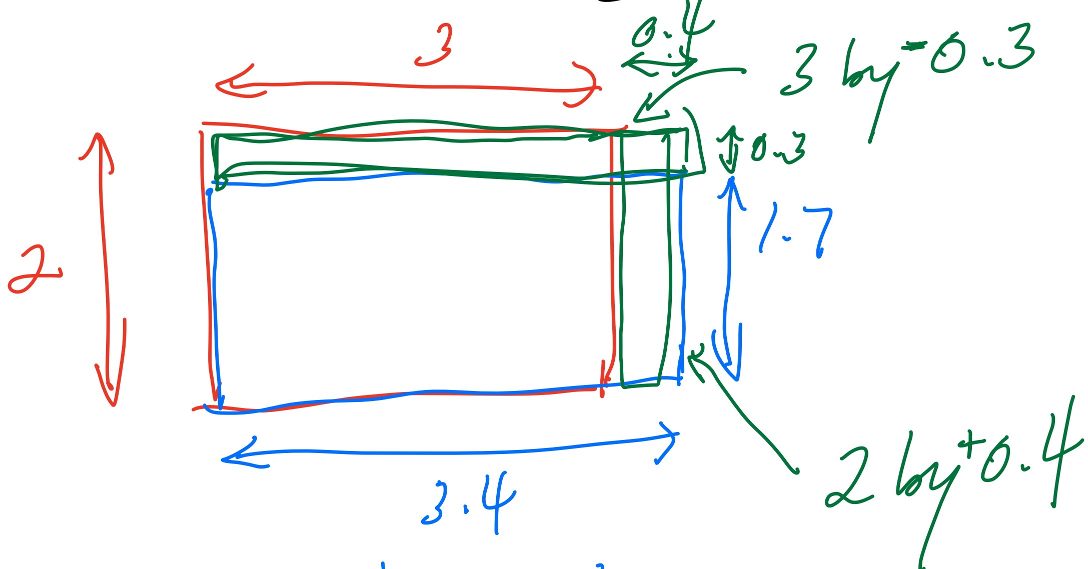
## Linear approximation

Estimate area of 3.4 by 1.7 box

3      2




Area of 3.4 by 1.7 box

$$\approx 3(2) + 2(0.4) - 3(0.3)$$

$$6 + 0.8 - 0.9 = 5.9$$

Exact answer $= 5.78$

$A(h,w) = hw$

$A(h,w) - A(h_0, w_0)$

$= hw - h_0 w_0 = hw - \overbrace{h_0 w + h_0 w - h_0 w_0}^{\text{?}}$

$= (h - h_0) w + h_0 (w - w_0)$

$= (h - h_0) w_0 + h_0 (w - w_0)$

$(h - h_0)(w - w_0)$

If $h - h_0$, $w - w_0$ are small

then $(h-h_0)(w-w_0)$ is really small

So $A(h,w) \simeq (h-h_0) w_0 + h_0 (w - w_0)$

Linear approximation

- Suppose $(x,y)$ is close to $(x_0,y_0)$
- $f(x,y)$ is a function where we know the value $f(x_0,y_0)$
- Estimate $f(x,y)$
- Zeroth estimate : $f(x,y) \approx f(x_0,y_0)$
- $1^{st}$ order estimate

$$f(x,y) \approx f(x_0,y_0) + \left(\begin{array}{c}\text{change in } f \text{ due to} \\ \text{change in } x\end{array}\right) + \left(\begin{array}{c}\text{change in } f \text{ due} \\ \text{to change in } y\end{array}\right)$$

$$\approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Linear approximation

Right side : $(x_0, y_0)$, $f(x_0, y_0)$, $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ are constants

$\Rightarrow$ Right side is a linear function of $x$ and $y$

$$f(x, y) \approx ax + by + c$$

$a = f_x(x_0, y_0)$ $\quad c = \ldots$

$b = f_y(x_0, y_0)$

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Example

$$f(x, y) = \sqrt{x^2 + y^2}$$

$$f_x = \frac{x}{\sqrt{x^2 + y^2}}, \quad f_y = \frac{y}{\sqrt{x^2 + y^2}}$$

$$f(3, 4) = 5$$

$$f(3.1, 3.9) \approx f(3,4) + f_x(3,4)(3.1 - 3) + f_y(3,4)(3.9 - 4)$$

$$= 5 + \frac{3}{5}(0.1) + \frac{4}{5}(-0.1)$$

$$= 5 + 0.6 - 0.8 = 4.8$$