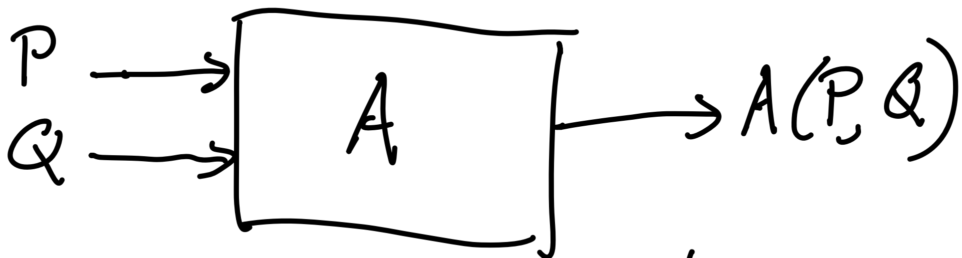
Functions of more than one variable



A is name of function

P, Q inputs to A

$$R = A(P, Q)$$

"Set R equal to output of A with P, Q as inputs"

Example

$$A(S,T) = S^2 + ST - T^2$$

$$A(2,3) = 2^2 + 2(3) - 3^2$$

$$= 4 + 6 - 9 = 1$$

$$A(S+T, T) = ?$$

$$A(P, Q) = P^2 + PQ - Q^2$$

$$A(S+T, T) = (S+T)^2 + (S+T)T - T^2$$

- Graph of function of 2 variables

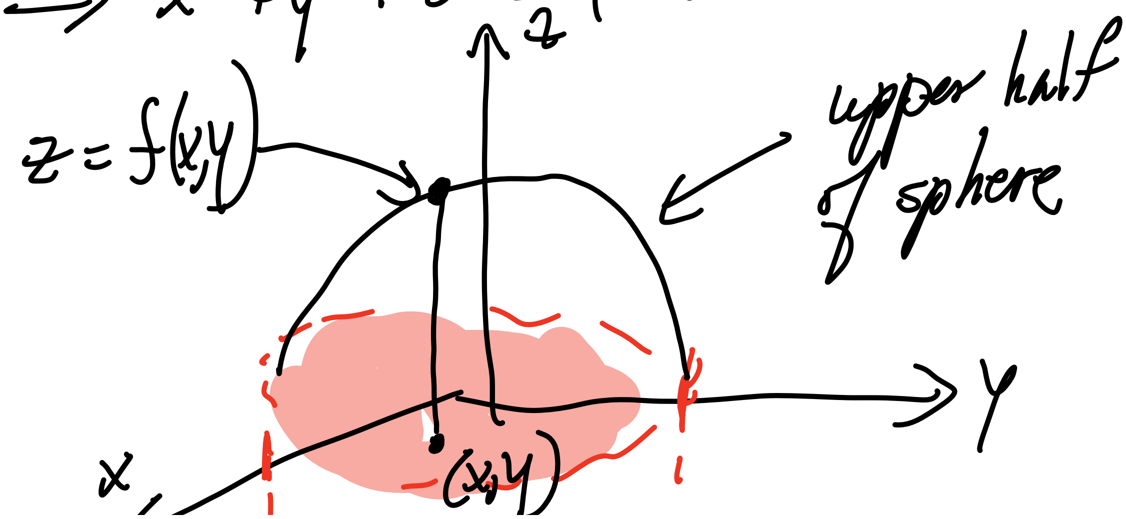
$$f(x,y) = \sqrt{1-x^2-y^2}$$

domain: $x^2+y^2 \leq 1$

$z = f(x,y)$ defines a surface in xyz-space

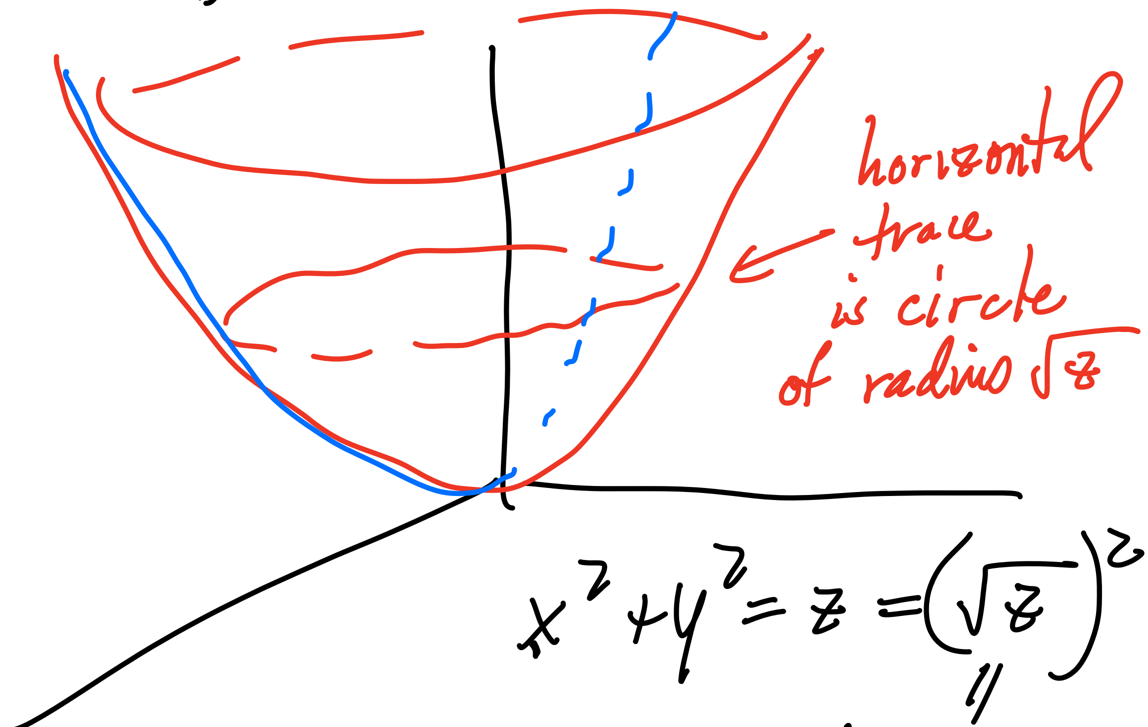
$$z = \sqrt{1-x^2+y^2}$$

$\Longrightarrow x^2+y^2+z^2 = 1$ and $z \geq 0$



Circular Paraboloid

$h(x,y) = x^2+y^2 \Rightarrow$ Graph is $z = x^2+y^2$



$x^2+y^2 = z = (\underbrace{\sqrt{z}}_{r})^2$

Vertical trace is parabola

If $y=0$, get $z=x^2$

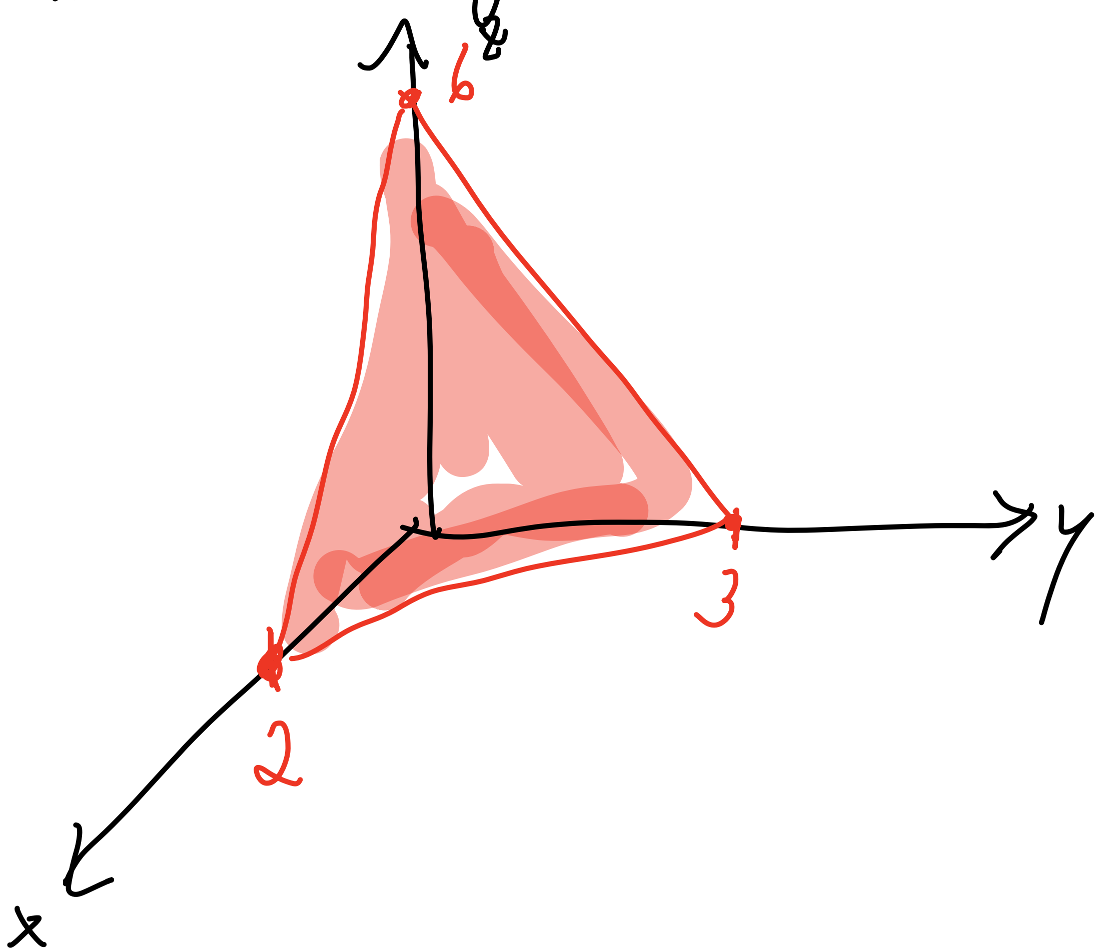
Graph of linear function

$$f(x,y) = 6 - 3x - 2y$$

is $\quad z = 6 - 3x - 2y$

or $\quad 3x + 2y + z = 6$

which is a plane



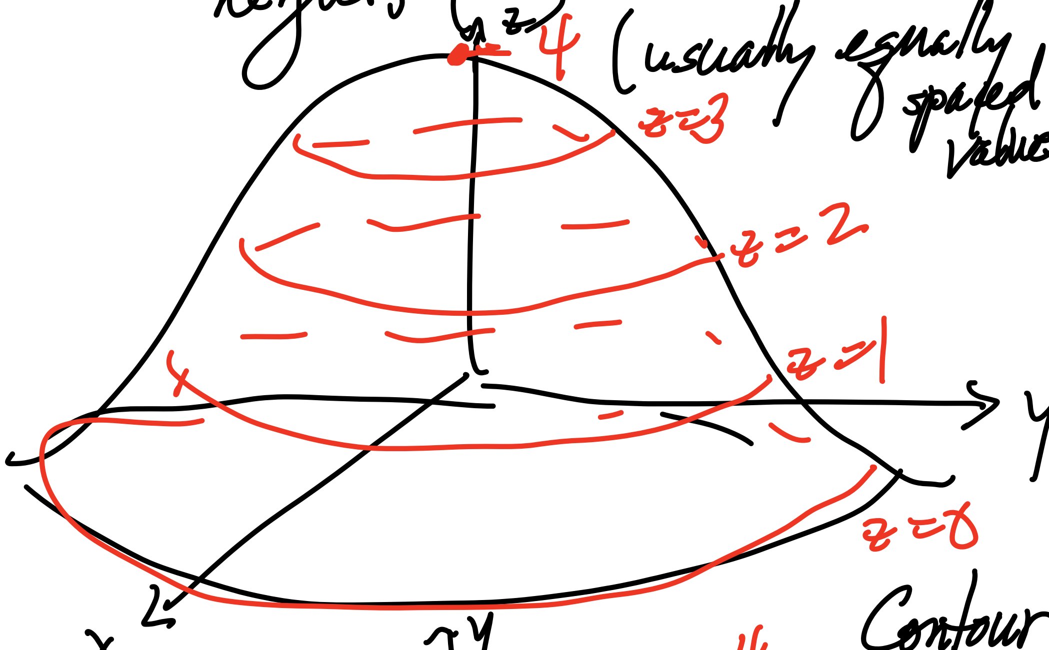
3 (main) ways to describe a function

1) Formula $f(x,y) = 3xy^2$

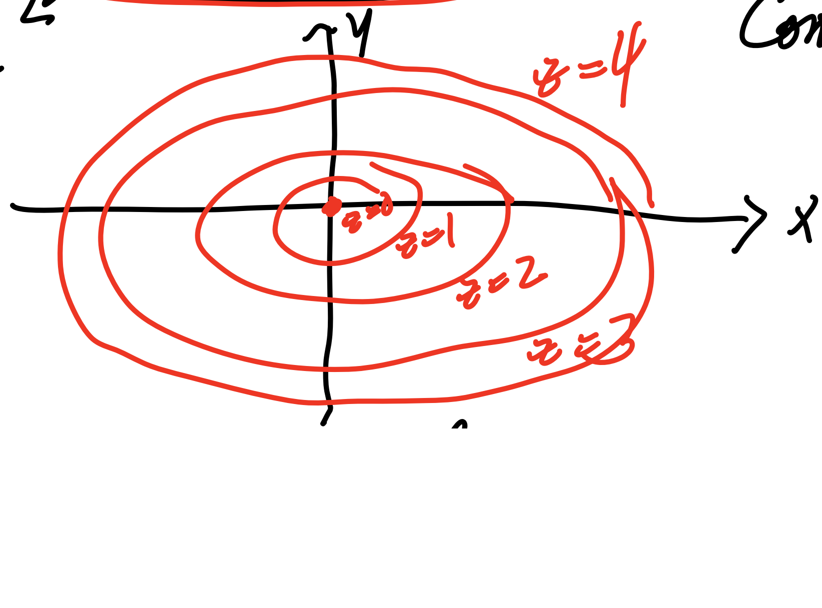
2) Graph $z = f(x,y)$

3) Contours or level curves

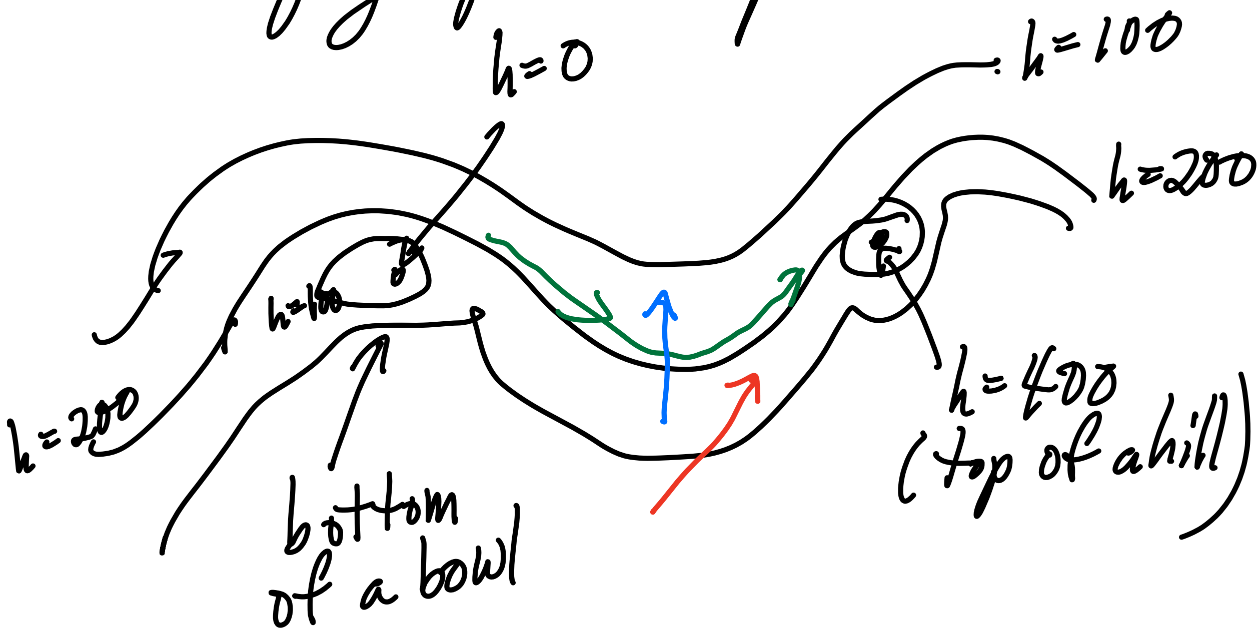
Horizontal traces at different heights (i.e. different values of z) (usually equally spaced values of z)



Contour map



Topographical map



Contour of linear function

$$f(x,y) = ax + by + c$$

Contour

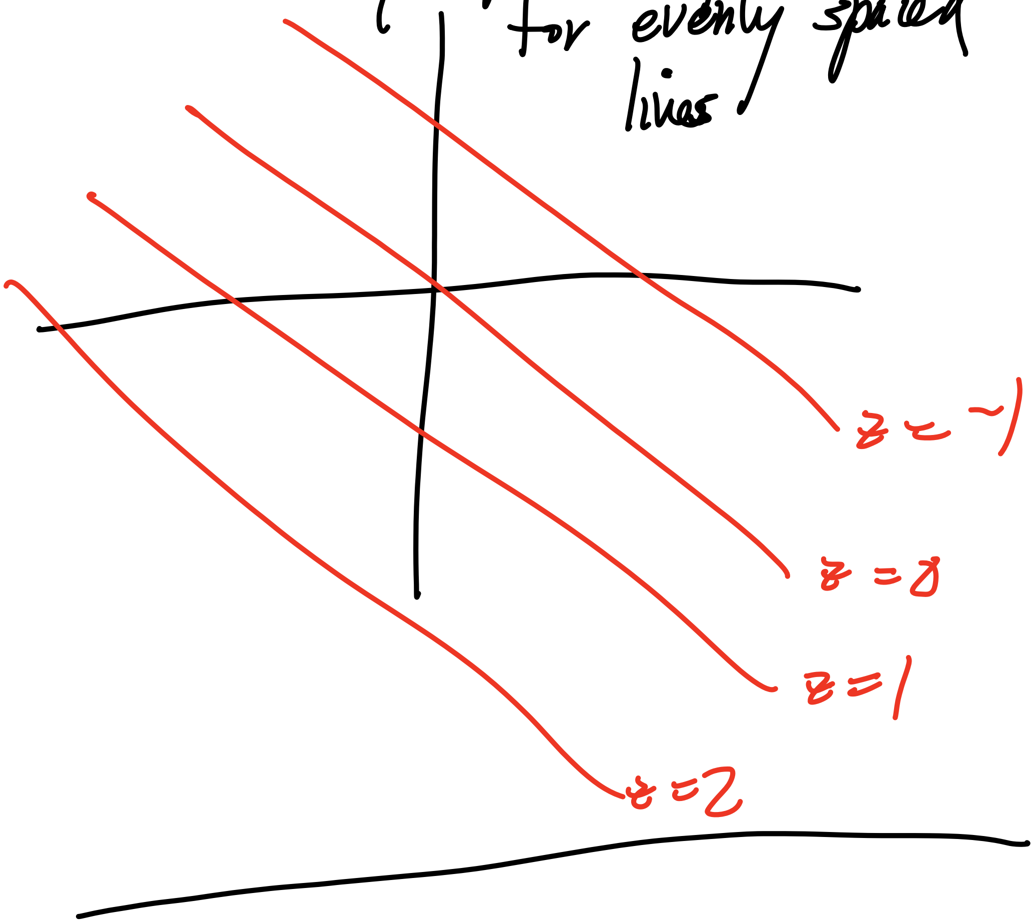
$$h = f(x,y) \Rightarrow ax + by + c = h$$

$$ax + by = h - c$$

$\Longrightarrow$ contours are parallel

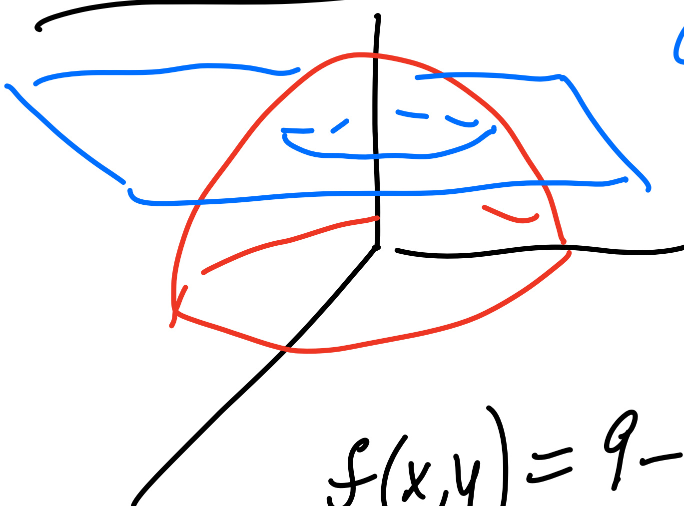
As h changes, line shifts upward by distance h

$\Longrightarrow$ For evenly spaced values of h, Get evenly spaced parallel lines for evenly spaced lines



Paraboloid

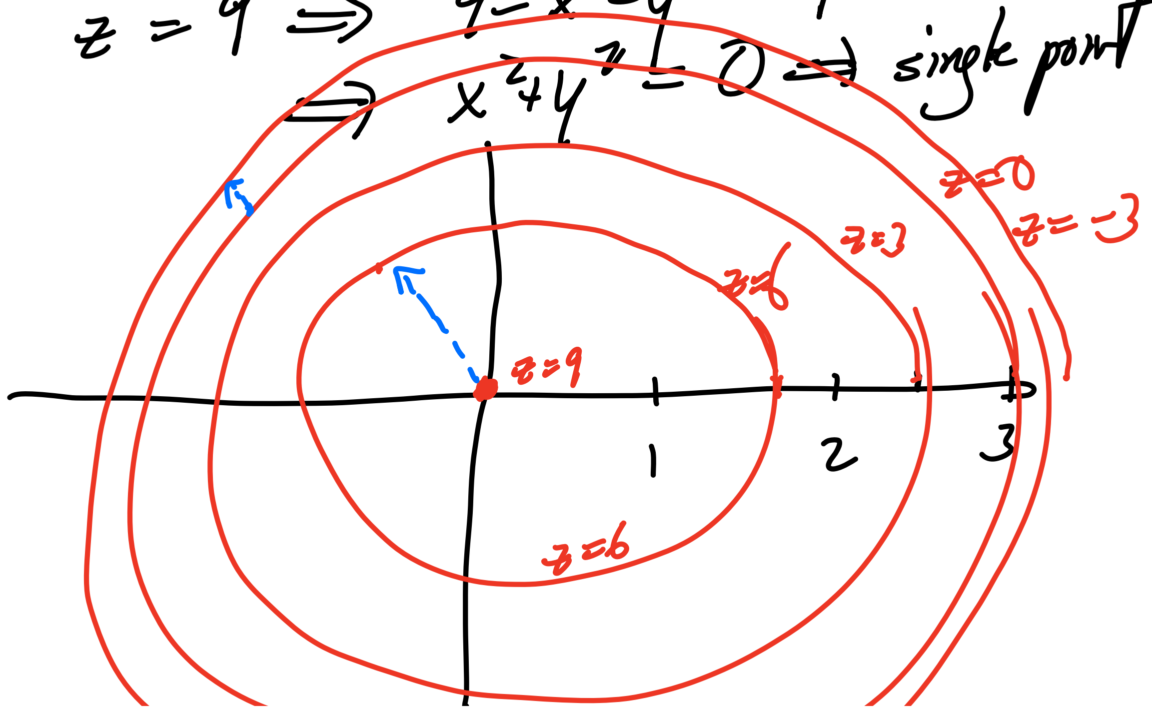
$$f(x,y) = 9 - x^2 - y^2,$$
$$x^2 + y^2 \leq 9$$



Contours are all circles

$$f(x,y) = 9 - x^2 - y^2$$

$$z = 9 \Rightarrow 9 - x^2 - y^2 = 9$$
$$\Rightarrow x^2 + y^2 = 0 \Rightarrow \text{single point}$$



$z=6 \Rightarrow 9-x^2-y^2=6$

$x^2+y^2=3=(\sqrt{3})^2 \approx (1.7)^2$

$z=3 \Rightarrow 9-x^2-y^2=3$

$x^2+y^2=6=(\sqrt{6})^2$

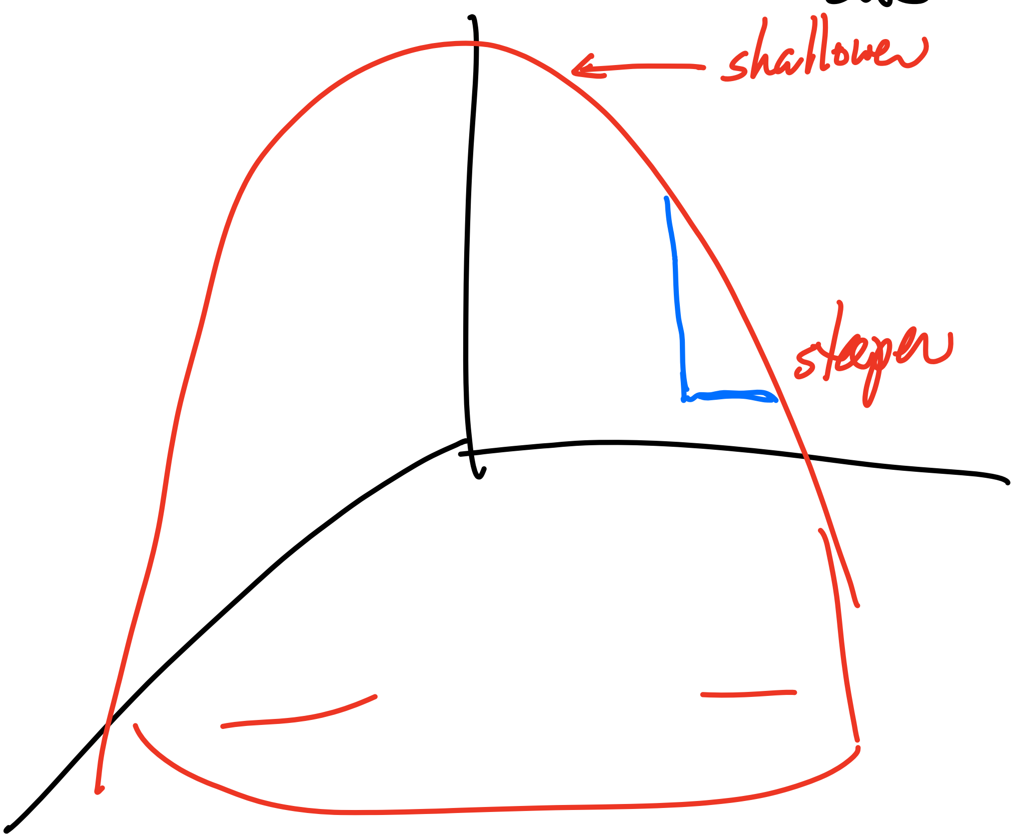
$=(2.5)^2$

$z=0 \Rightarrow 9-x^2-y^2=0$

$x^2+y^2=9=3^2$

$z=-3 \Rightarrow x^2+y^2=12=(\sqrt{12})^2$

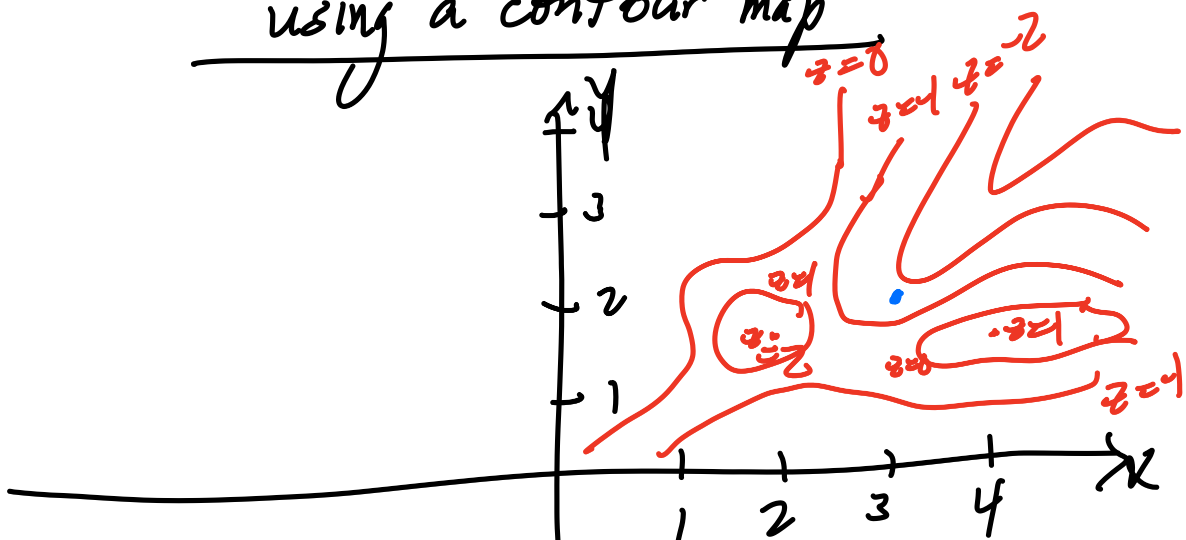
$=2\sqrt{3}$



Closely spaced contours
(for evenly spaced values of z)
$\Longrightarrow$ steep surface

Widely spaced contours
$\Longrightarrow$ shallow surface

Estimate value of function using a contour map



$f(3,2) = ?$

between $z=-1$, $z=-2$

closer to $z=-2$

$\Rightarrow f(3,2) \approx 1.7$

Function of 3 variables

$$f(x,y,z) = xy + y^2 z - 3x$$

Descriptions

1) Formula

2) ~~Graph is in 4-space~~

3) Contours

Example $f(x,y,z) = x^2+y^2+z^2$

$f=1$: $x^2+y^2+z^2=1$

$\Leftrightarrow$ sphere of radius 1

In general, contours of $f(x,y,z)$ are surfaces in 3-space

Can draw pictures in principle

Examples

1) Linear function

$$f(x,y,z) = ax + by + cz + d$$

Contour : $ax + by + cz + d = h$

$$ax + by + cz = \underbrace{h - d}_{\text{constant}}$$

$\Rightarrow$ Every contour has same normal direction $\langle a, b, c \rangle$

$\Rightarrow$ Parallel planes

$\Rightarrow$ Evenly spaced values of h give evenly spaced parallel planes

$$f(x,y,z) = x^2 + y^2 + 4z$$

Contour: $x^2 + y^2 + 4z = h$

$$z = \frac{h}{4} - \frac{1}{4}x^2 - \frac{1}{4}y^2$$

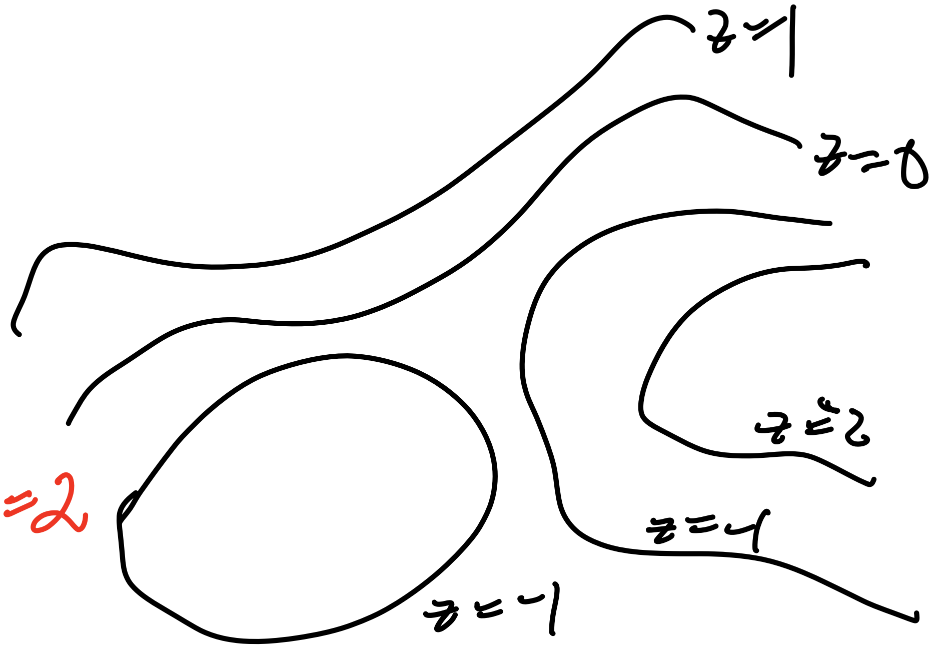
$h=0 \Rightarrow z = -\frac{1}{4}(x^2+y^2)$

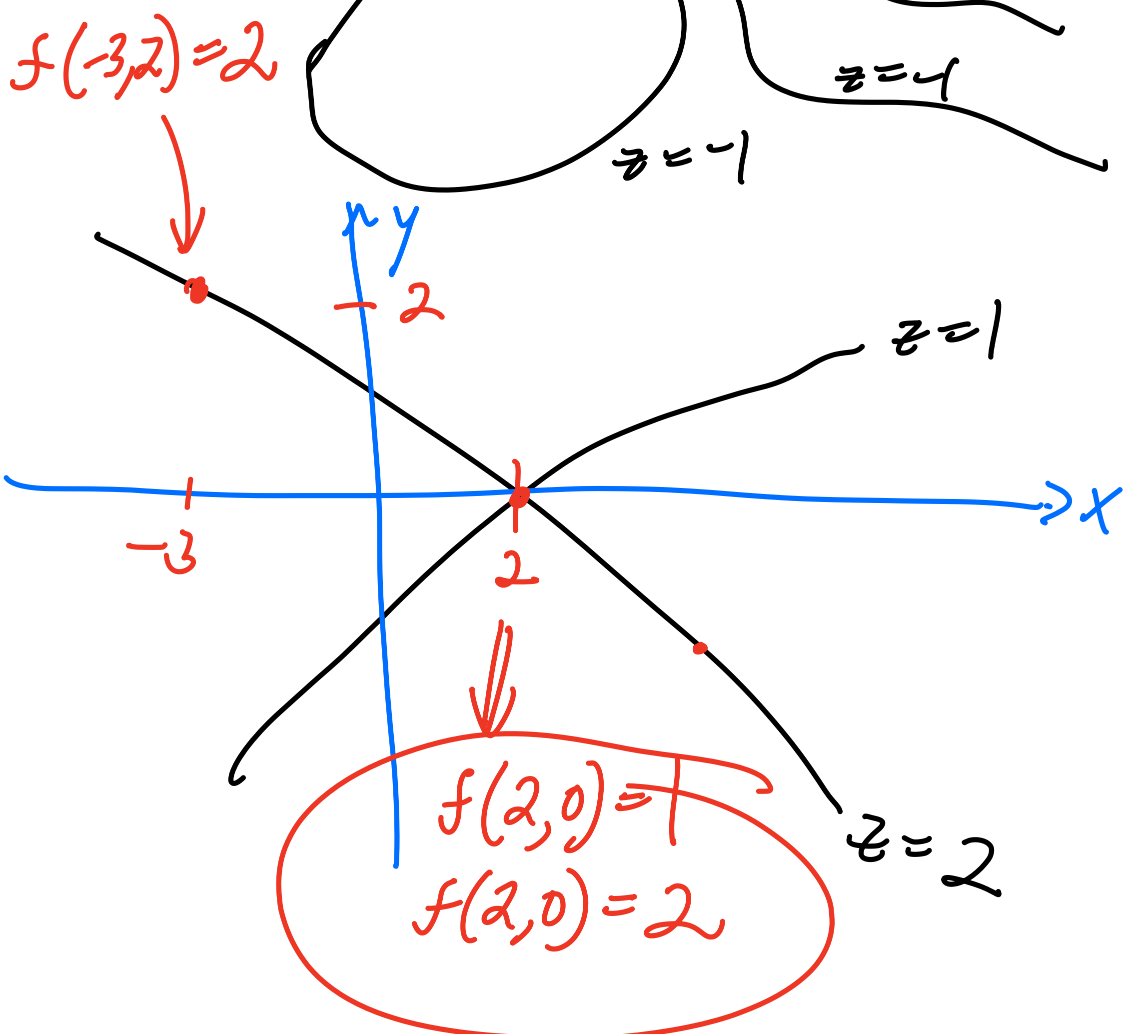
$h=1 \Rightarrow z = \frac{1}{4} - \frac{1}{4}(x^2+y^2)$

$h=2 \Rightarrow z = \frac{1}{2} - \frac{1}{4}(x^2+y^2)$

$\Longrightarrow$ evenly spaced copies of an upside down paraboloid

Same surface shifted vertically



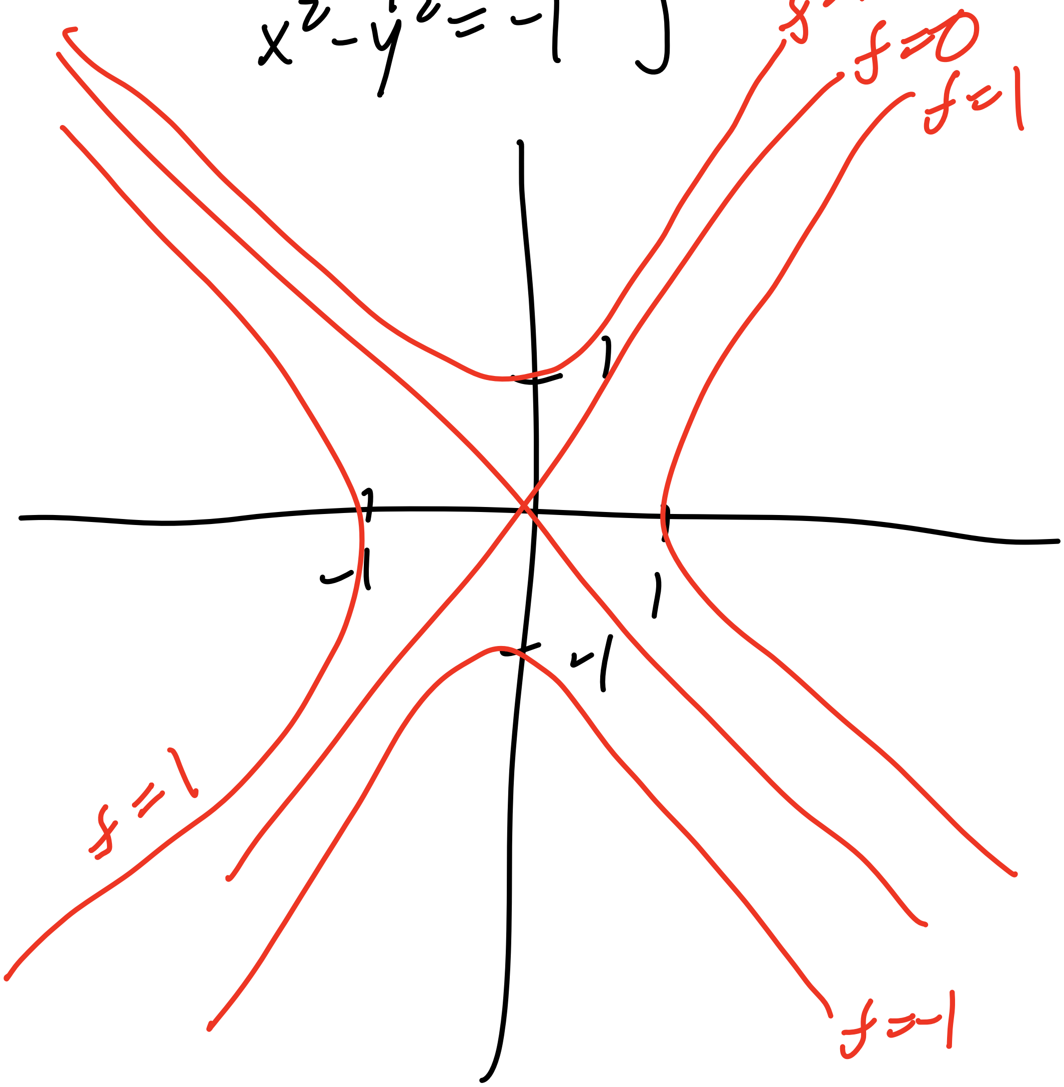


Impossible

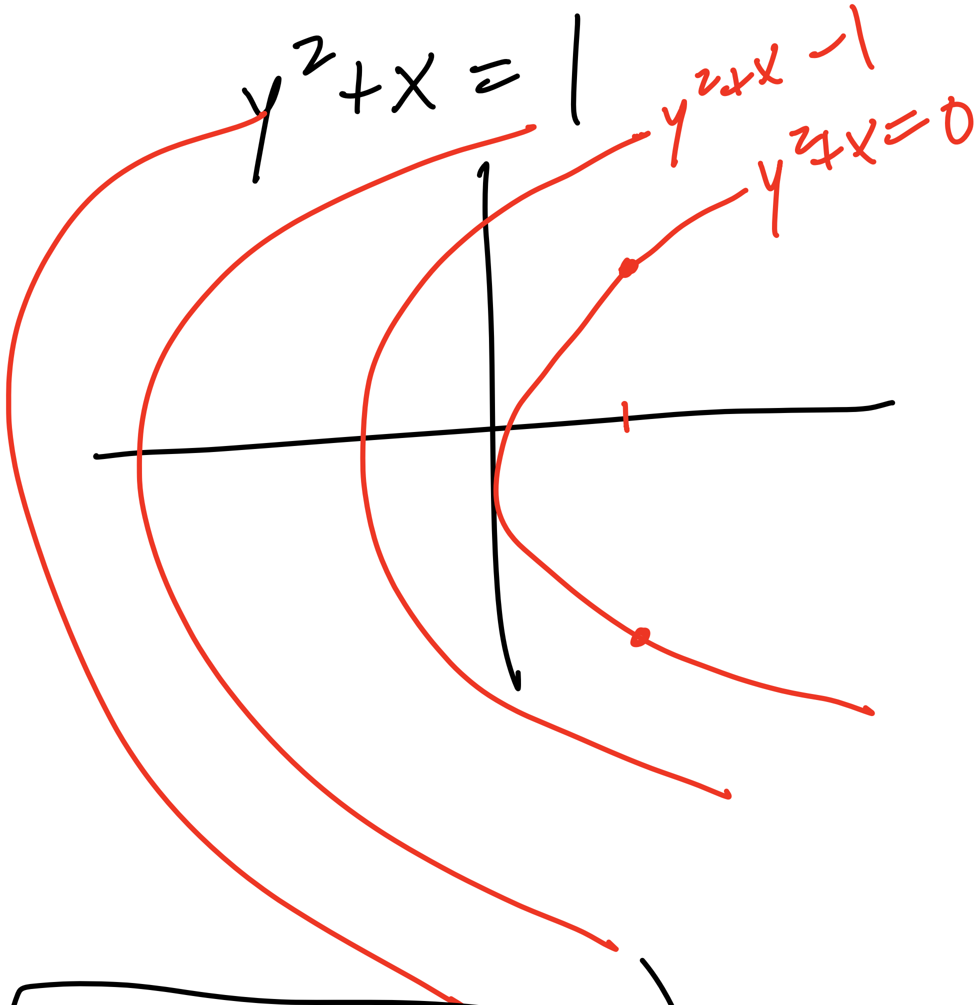
$\Longrightarrow$ Contour map is impossible

$$f(x,y) = x^2 - y^2$$

$$\left.\begin{array}{l} x^2 - y^2 = 1 \\ x^2 - y^2 = 0 \\ x^2 - y^2 = -1 \end{array}\right\} \text{hyperbolas}$$



$$f(x,y) = y^2 + x$$

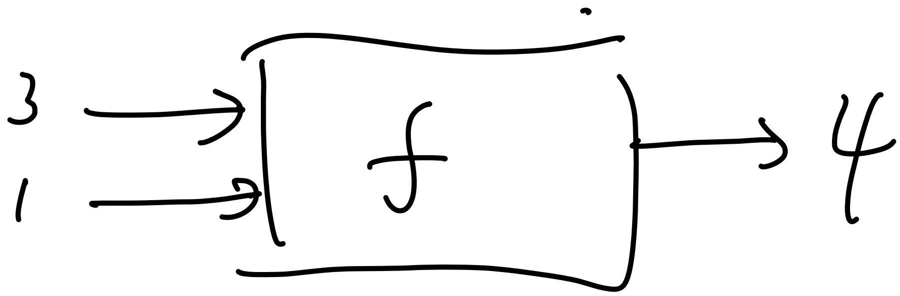


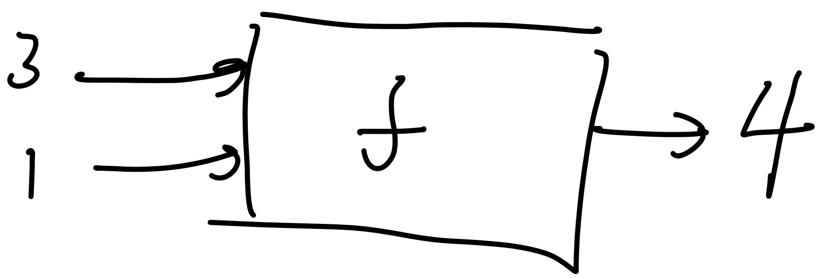
$$\boxed{4 = f(3,1) = 5}$$

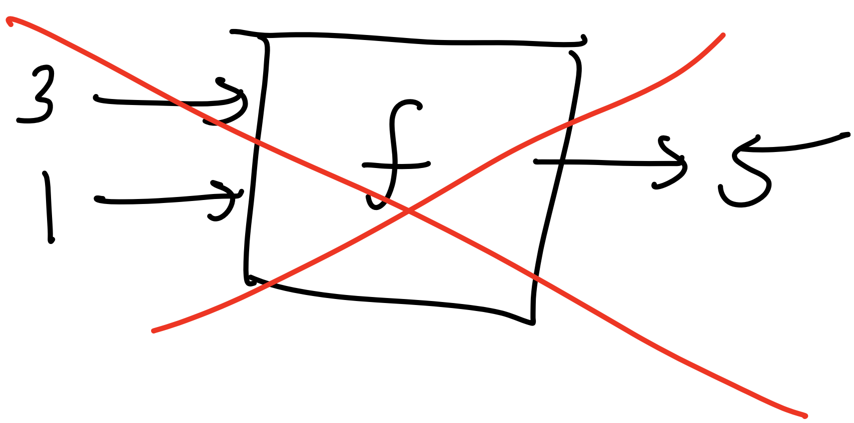
$$f(x,y,z) = x+y+z$$

$$f=1 \Rightarrow x+y+z=1$$

$$z=0 \Rightarrow x+y=1$$



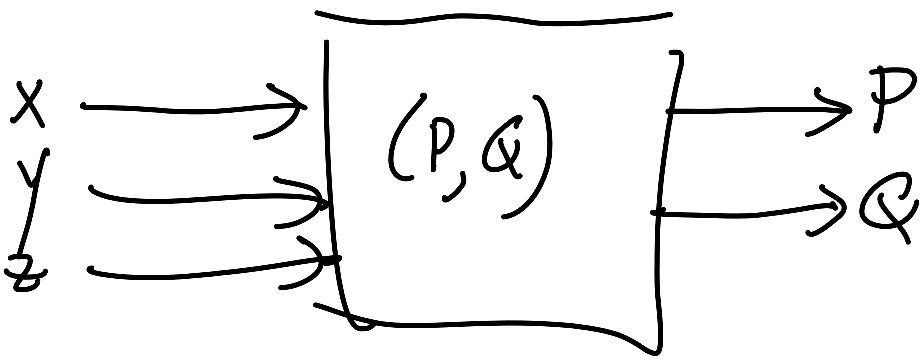




$$f(x,y,z) = x^2 + \sin y - e^{xz}$$

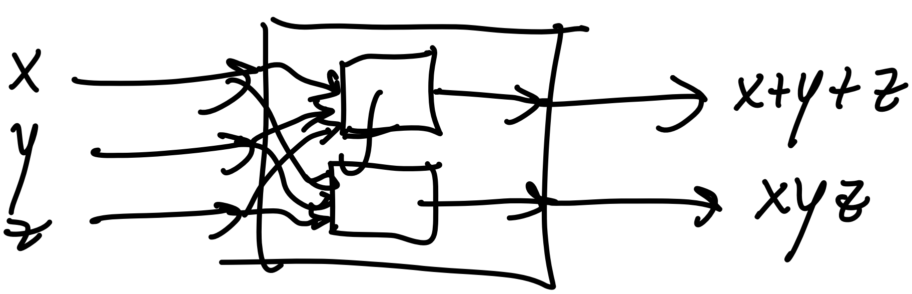
$$f(1,2,3) =$$

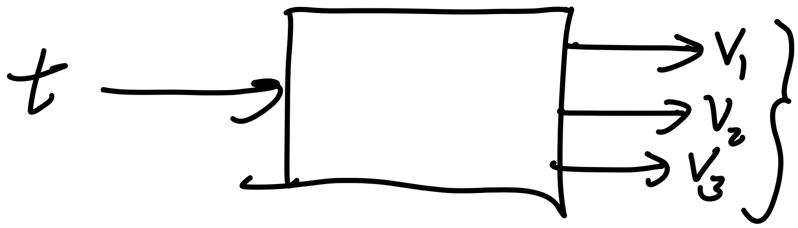
$$\cancel{f(1) = 3 \text{ and } f(1) = 5}$$



Input: (x,y,z)

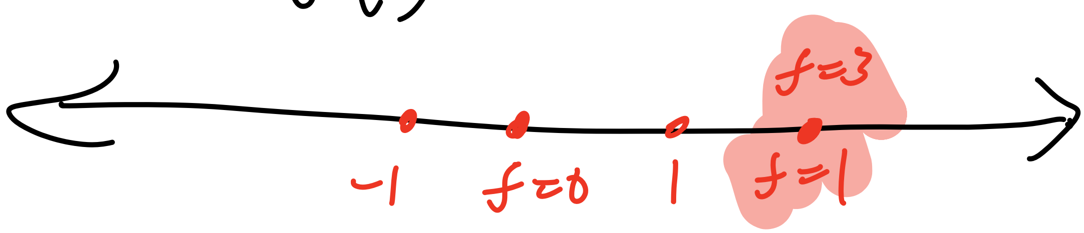
Output: $(P(x,y,z), Q(x,y,z))$





$\vec{V} = \langle V_1, V_2, V_3 \rangle$

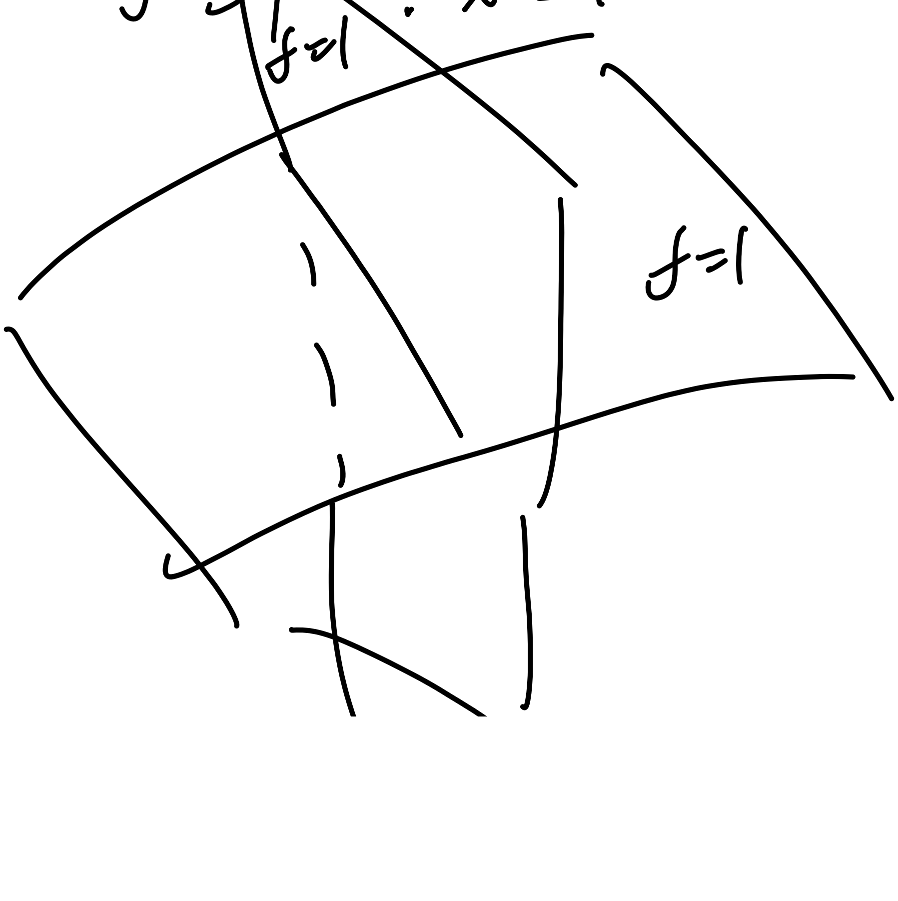
$$f(x) = x^2$$

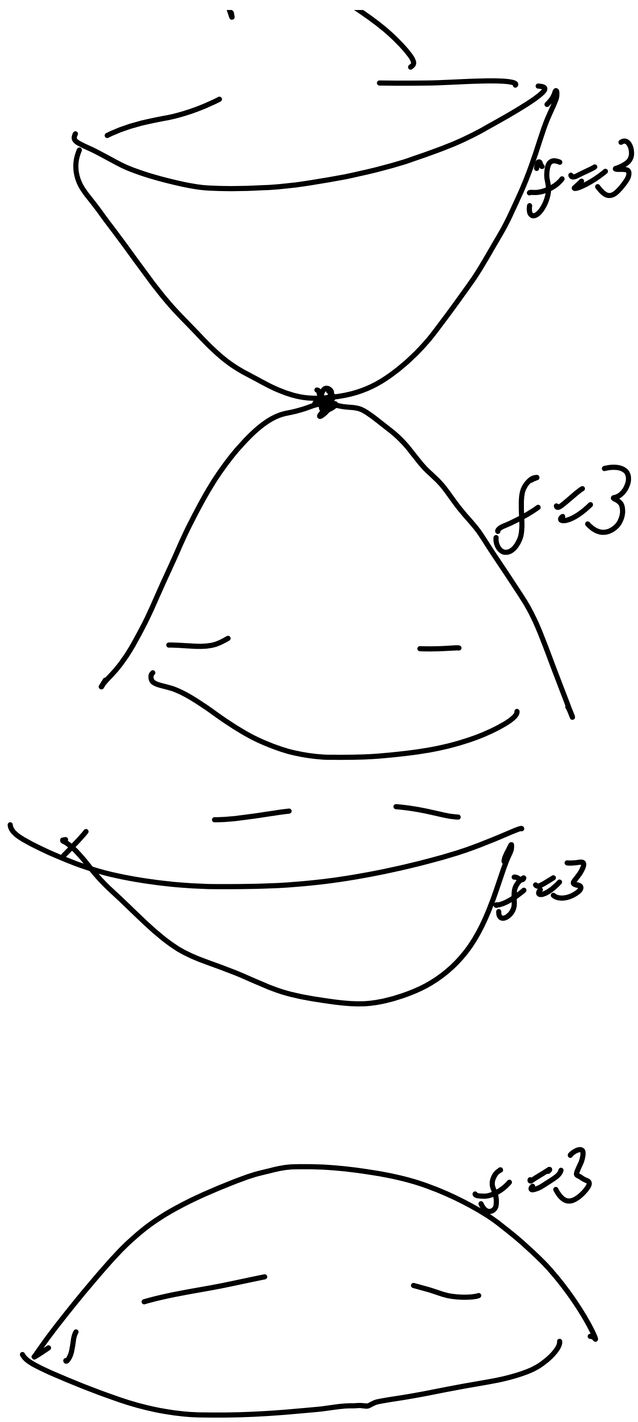


$f = -1 \;:\; x^2 = -1 \Rightarrow$ No contour

$f = 0 \;:\; x^2 = 0$

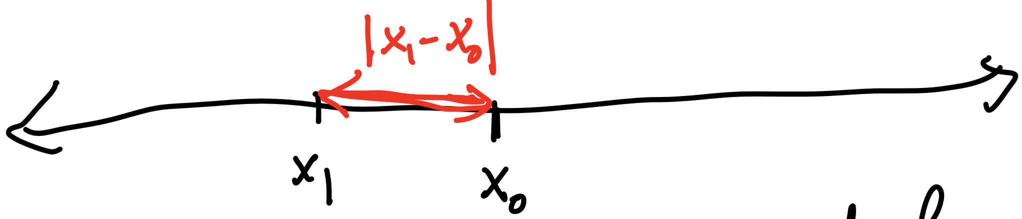
$f = 1 \;:\; x^2 = 1 \Rightarrow x = 1$ or -1




f=3
f=3
f=3
f=3

Margin of error and limit

Measurements



If x_1 is a real measurement of x_0
then error is $|x_1 - x_0|$

$\boxed{1m \pm .1m}$

Suppose m = margin of error > 0
x_1 is within margin of error
if $|x_1 - x_0| < m$

$x_1, x_2, \ldots\ldots$ infinite sequence of values (numbers, points, or vectors)

$$\boxed{\lim_{k \to \infty} x_k = L}$$

$$|x-y| = \begin{cases} \text{absolute value if } x,y \text{ numbers} \\ \text{distance from } x \text{ to } y \text{ if } x,y \text{ points or vectors} \end{cases}$$

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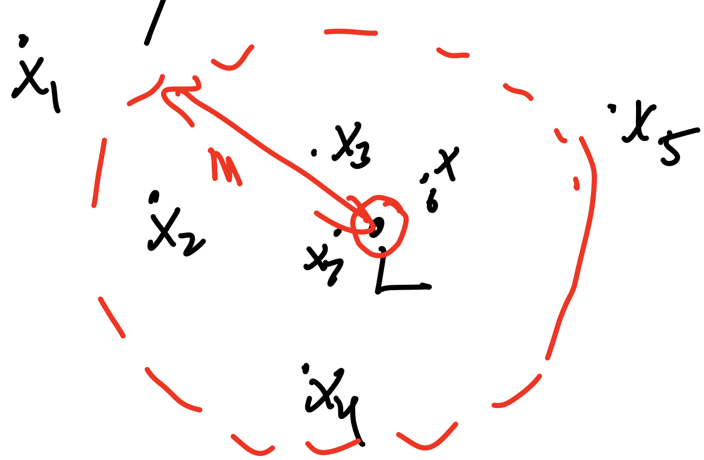
1) View each x_n as a measurement

2) Choose a margin of error $m > 0$

Want <u>all</u> measurements beyond a certain point in sequence to be within the margin of error

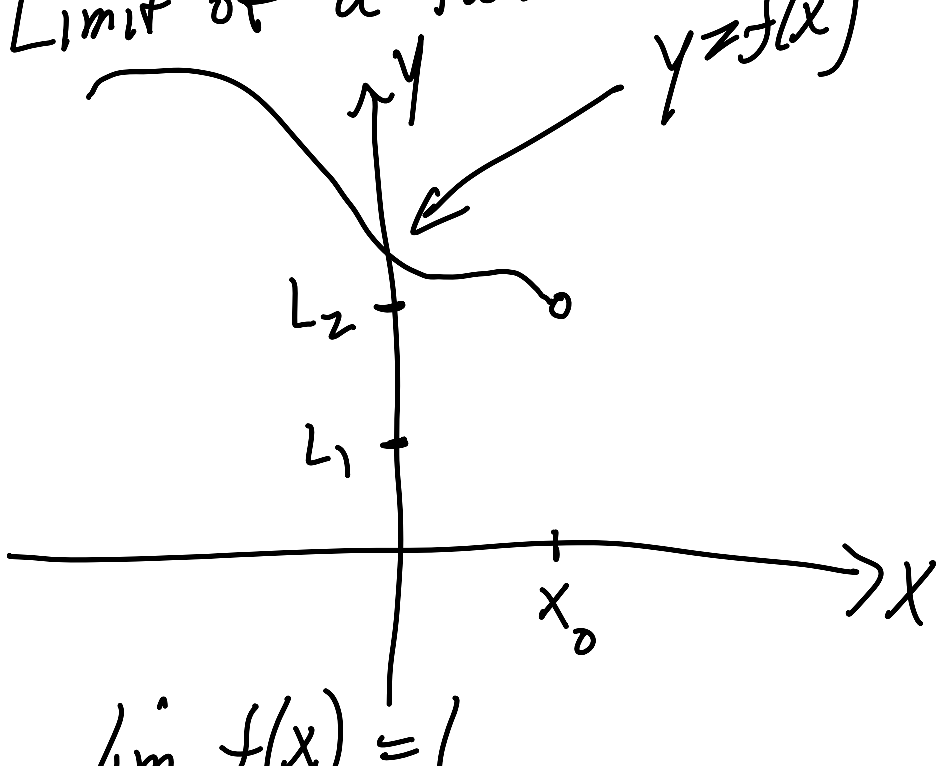
3) Want this to be true for any $m > 0$, no matter small



Precise version of $\lim_{n\to\infty} x_n = L$

Given any margin of error $m > 0$,
there is an $N_m > 0$
such that
$$|x_n - L| < m$$
for every $n > N_m$

Limit of a function



$$\lim_{x\to x_0} f(x) = L$$

Margin of error for output of f

Margin of error for input to f

Idea: Choose a margin of error for output, $\varepsilon > 0$

Find a margin of error $\delta > 0$ for input of f such that if $|x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$

$\lim_{x \to x_0} f(x) = L$ means no matter how small $\varepsilon > 0$ is, there is a $\delta > 0$ s.t. if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$

Another way

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

is true if for any sequence of inputs such that

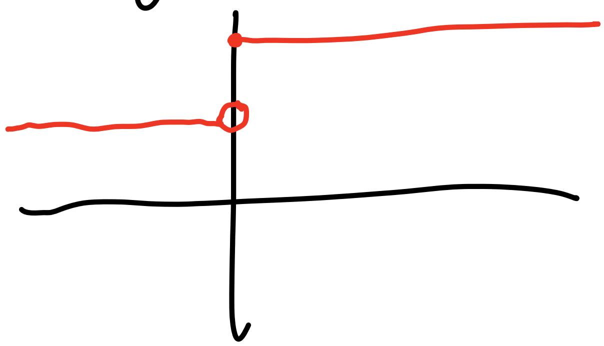
$$\lim_{k\to\infty} (x_k, y_k) = (x_0, y_0),$$

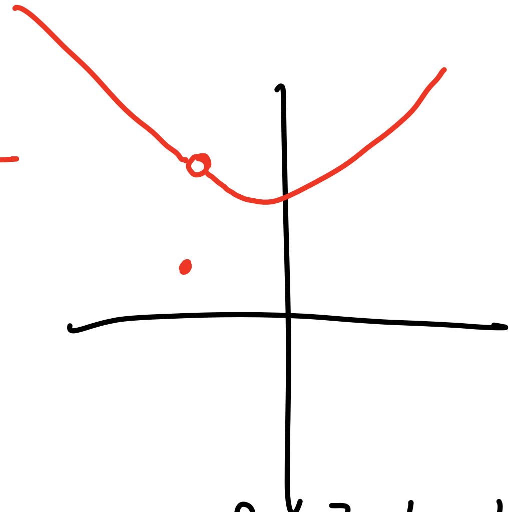
then

$$\lim_{k\to\infty} f(x_k, y_k) = L$$

Continuity

f is continuous if it never jumps in value





$$f(x) = \begin{cases} x^2+1 & x \neq -1 \\ \frac{1}{2} & x = -1 \end{cases}$$

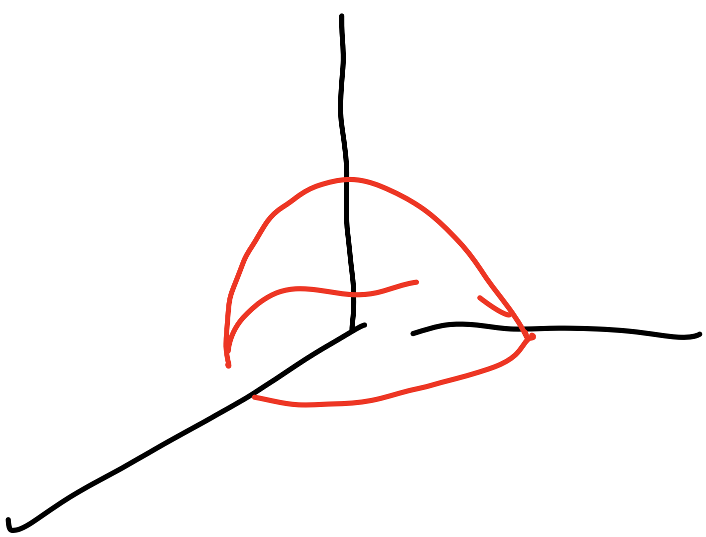
$f(x,y)$ continuous at (x_0, y_0)

if

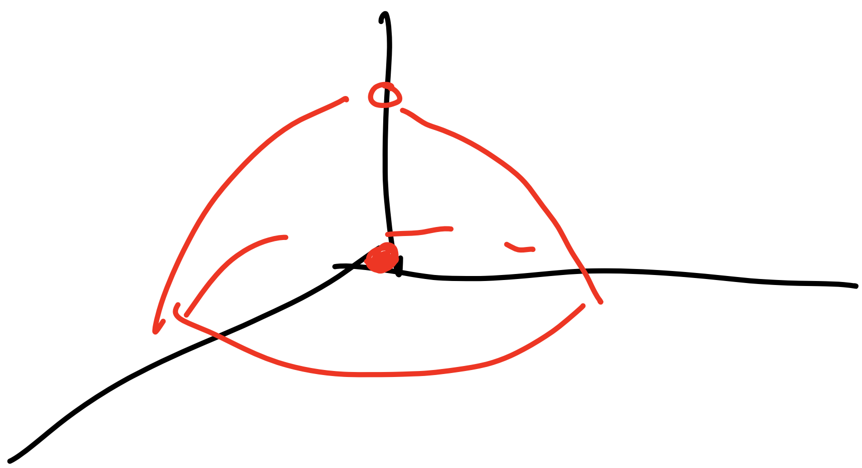
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

f is continuous if its graph is a continuous surface (no jumps or tears

$$f(x,y) = \sqrt{1-x^2-y^2}$$



$$f(x,y) = \begin{cases} \sqrt{1-x^2-y^2} & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = 0 \end{cases}$$



<u>General rule</u> (Informal)

If f is defined using a single formula, then f is continuous at every point in its <u>domain</u>

$f(x,y) = \sqrt{1-x^2-y^2}$

is continuous if $x^2+y^2 \leq 1$

$$f(x,y) = \frac{x+y}{x-y}$$

undefined if $x=y$

domain is $\{(x,y) : x \neq y\}$

f is continuous on this domain

$$f(x,y) = \begin{cases} \dfrac{2x^2y-5xy^2}{x^2+3y^2} & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y)=0 \end{cases}$$

- f is continuous if $(x,y) \neq 0$
- $\boxed{f \text{ is continuous at } (0,0) \text{ if } \lim_{(x,y)\to(0,0)} f(x,y) = 0}$

Examples

1) $$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy + 1}{y^2 - 3} = \frac{-1}{3}$$

Try plugging in $(0,0)$

$$\frac{0 - 0 + 1}{0 - 3} = \frac{-1}{3}$$

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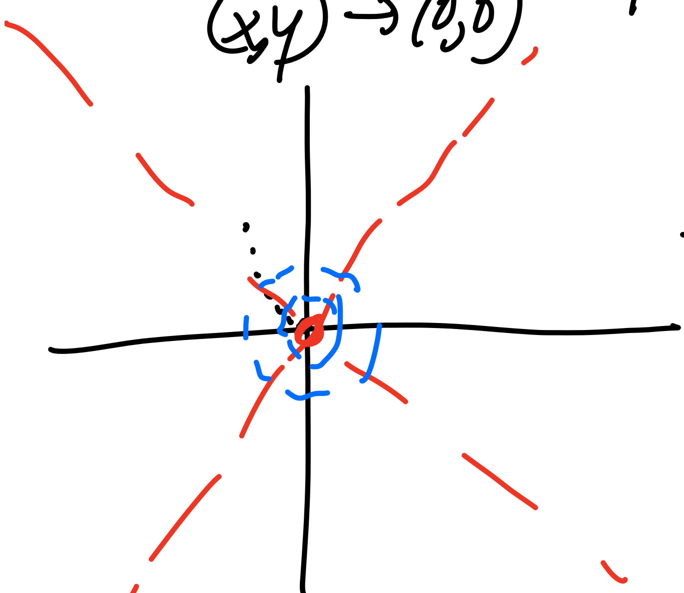
2) $$\lim_{(x,y)\to(0,0)} \frac{x^2 + 2y}{x^2 - y^2}$$



When is denominator zero?

$x^2 - y^2 = 0$

Formula blows up along $y = \pm x$ because $\frac{\text{nonzero}}{\text{zero}}$

There is a sequence $(x_n, y_n) \to (0,0)$ where formula is always undefined $\Rightarrow$ limit does not exist