$\vec{k} = \hat{k} = \langle 0,0,1 \rangle$

$\vec{j} = \langle 0,1,0 \rangle$

$\vec{i} = \langle 1,0,0 \rangle$

$\vec{b} = \vec{k} \times \vec{a}$

$\vec{a}$ is given $\vec{a} = a_1\vec{i} + a_2\vec{j} + 0\vec{k}$

What are all possible vectors $\vec{b}$

such that $\vec{b} = \vec{k} \times \vec{a}$ (*)

Suggestion: $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

Substitute this into *

and find all possible solutions for $\vec{b}$ (in terms of a_1, a_2)

Or use picture (geometric description of $\vec{a}, \vec{b}, \vec{k}$ and cross product

$$(x,y,z) = (t, t-1, t)$$
$$\begin{array}{ccc} \shortparallel & \shortparallel & \shortparallel \\ x & y & z \end{array}$$

Does this point lie on the planes

$$x - 2y + z = 2$$
$$x - z = 0$$

Test

$$x - 2y + z = t - 2(t-1) + t$$
$$= t - 2t + 2 + t$$
$$= 2$$

For any t
$(t, t-1, t)$ satisfies $x - 2y + z = 2$
and therefore lies in that plane

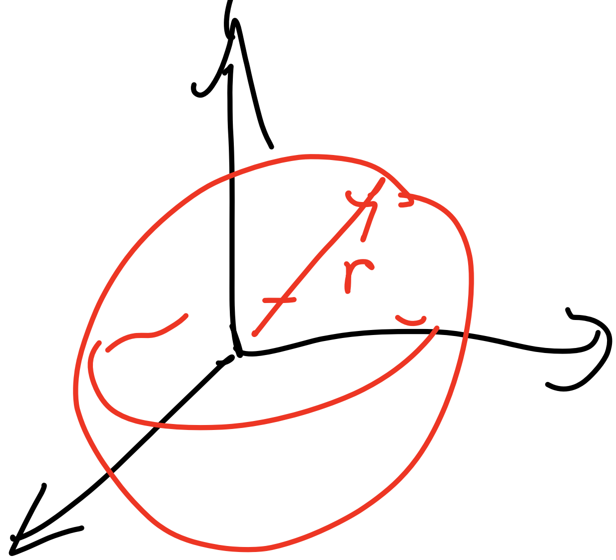
$$x - z = t - t = 0$$
$\Rightarrow$ lies in 2nd plane

Quadric Surfaces in 3-space

Described by an equation with a degree 2 polynomial in x, y, z

Example:

$$x^2+y^2+z^2=r^2$$

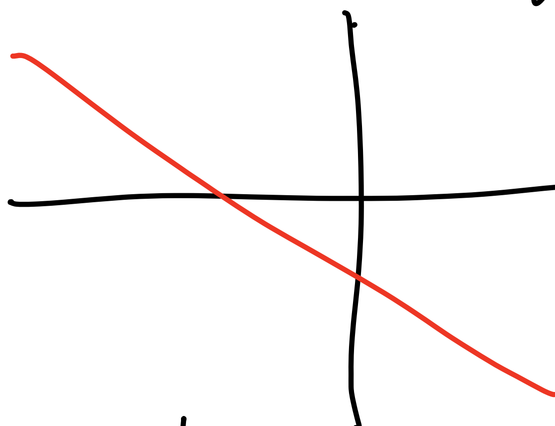


Review quadric curves in 2-space

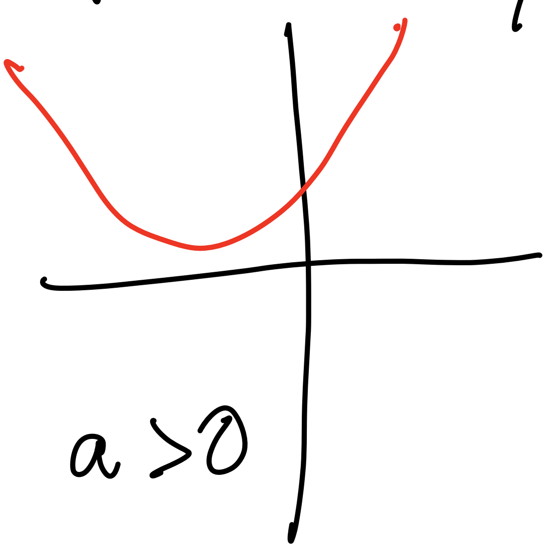
$$ax^2 + bxy + cy^2 + dx + ey = f$$

Line: $ax + by = c$

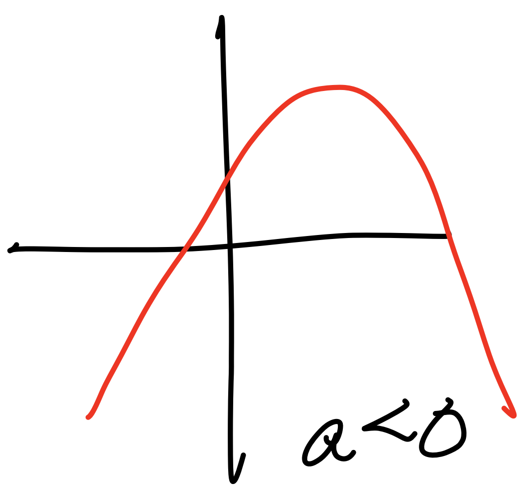
$$dx + ey = f$$



Parabola: $y = ax^2 + bx + c$

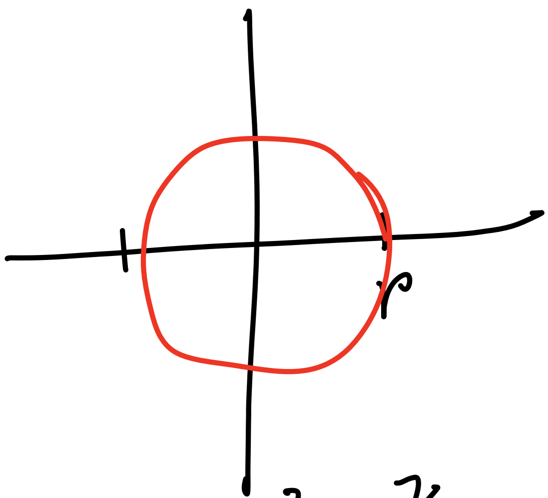


$a > 0$

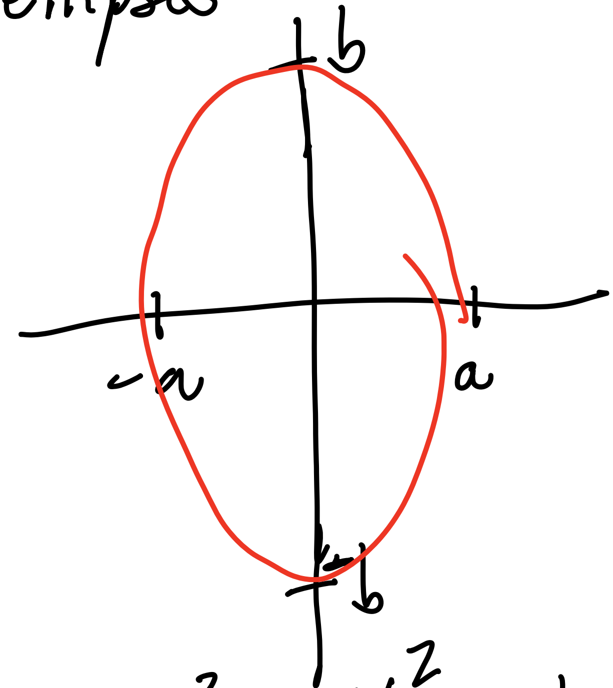


$a < 0$

Circles and ellipses



$$x^2+y^2=r^2$$



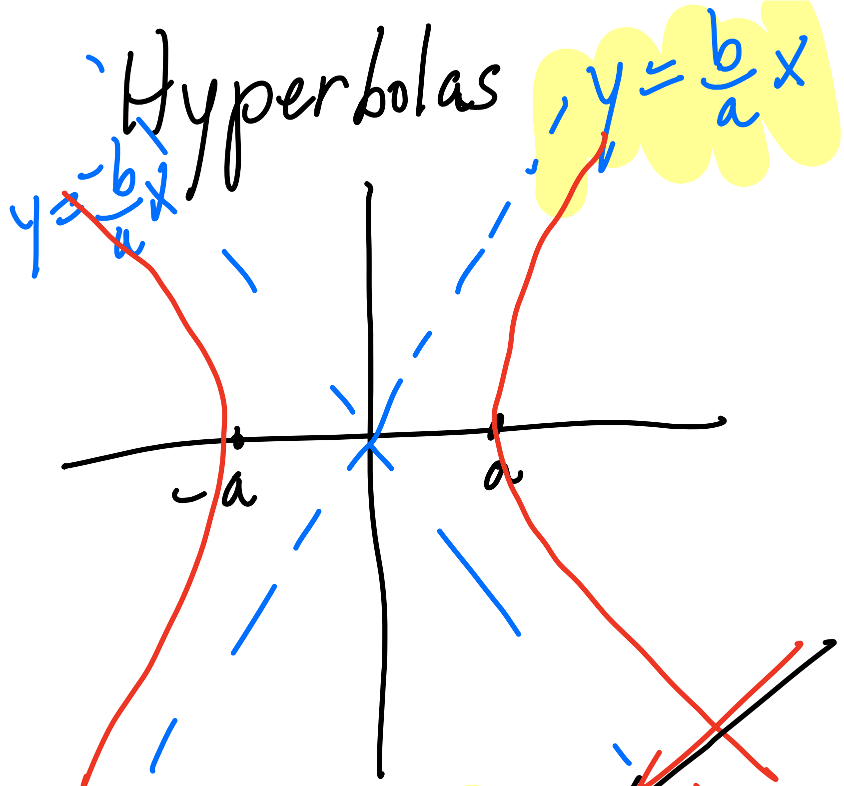
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Easy to see

$(\pm a,0)$, $(0,\pm b)$

lie on ellipse

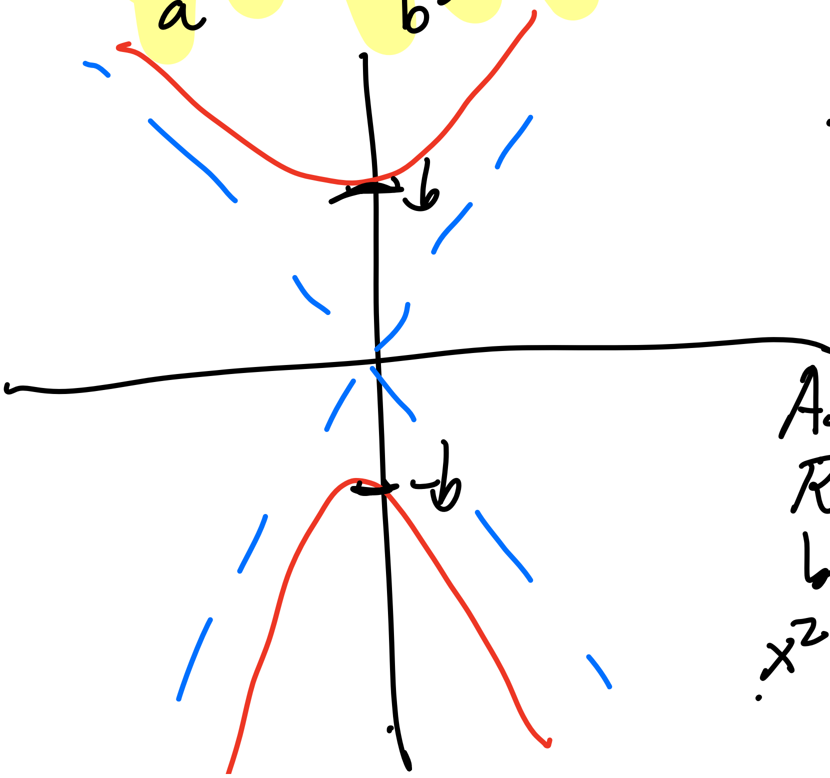
Hyperbolas



$(a, 0)$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



Asymptotes:
Replace ± 1
by 0

x^2

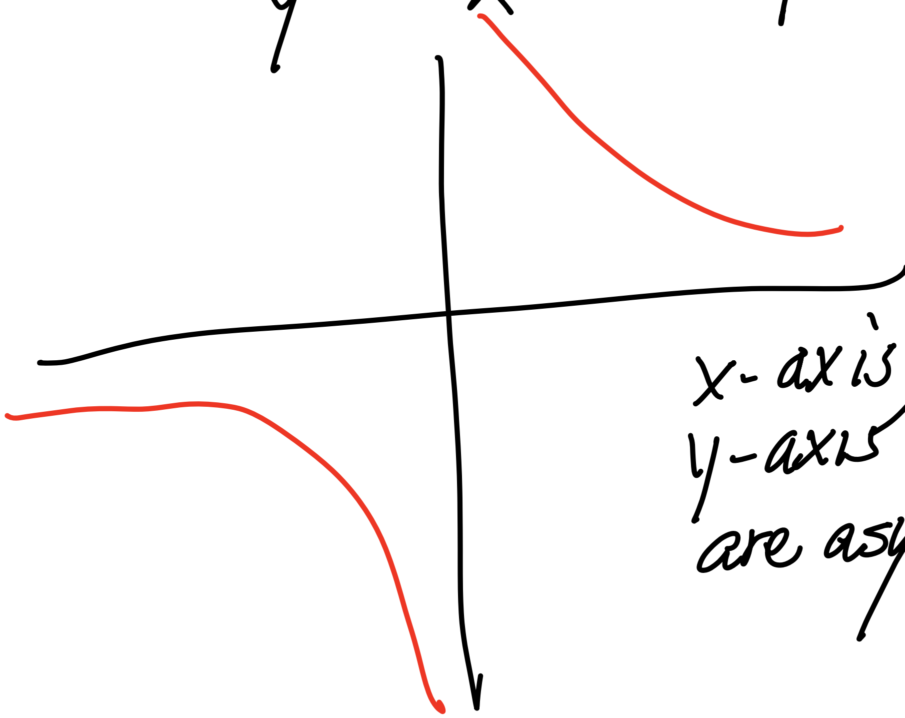
Asymptotes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 0$$

$$y = \pm \frac{b}{a} x$$

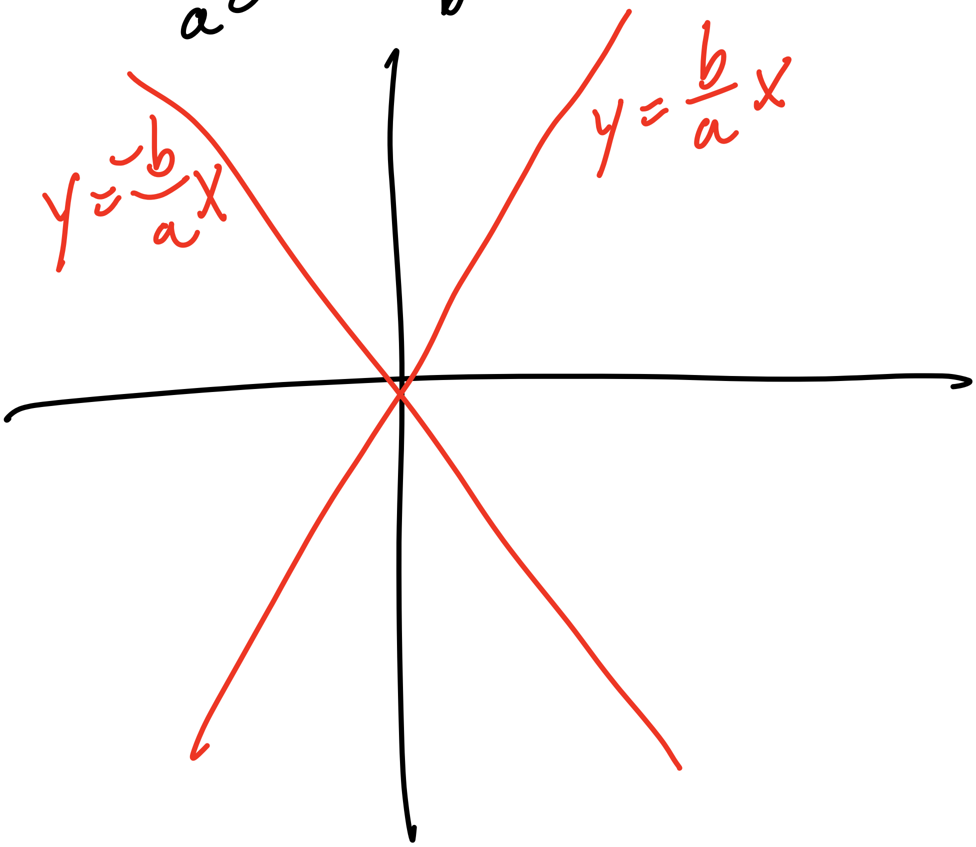
$$y = \frac{a}{x} \text{ or } xy = a$$



x-axis, y-axis are asymptotes

Crossing lines

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$



Quadric Surfaces

1) Plane: $ax + by + cz = d$

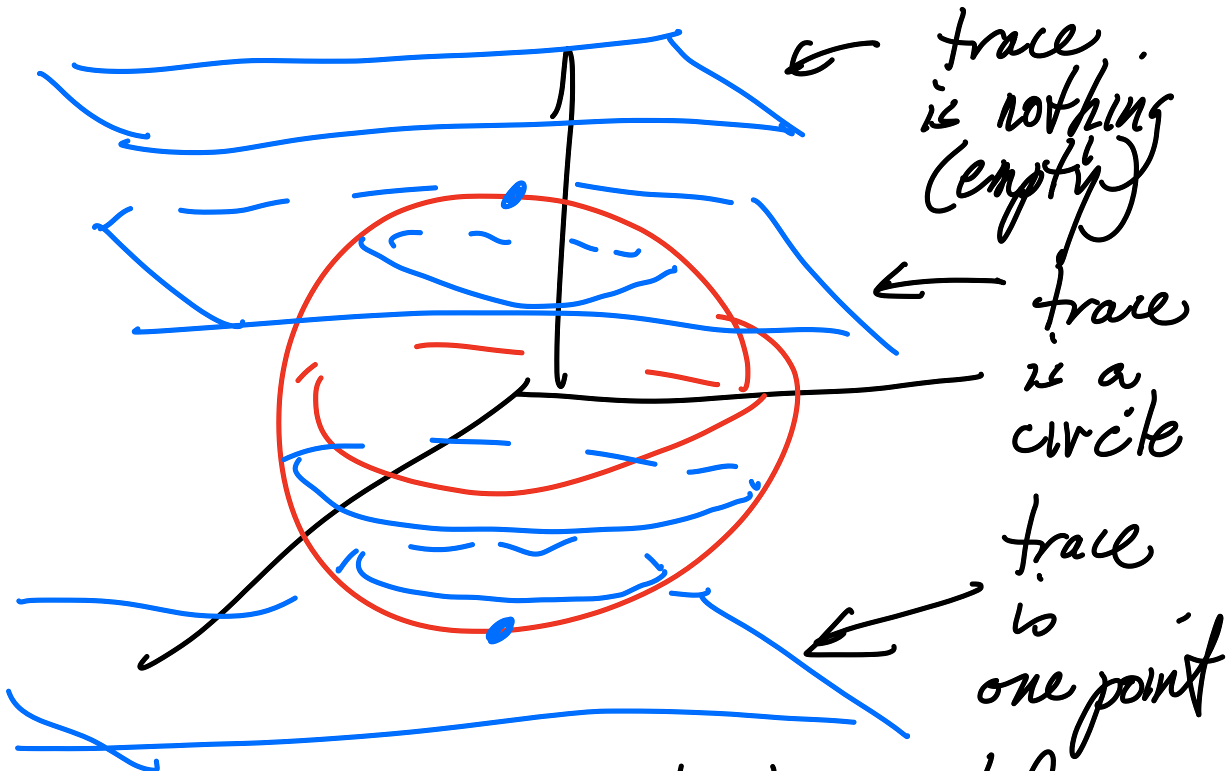
Traces of a surface

Idea: Intersection of surface with a plane, is usually one of following:

1) Nothing,
2) Points,
3) Curve

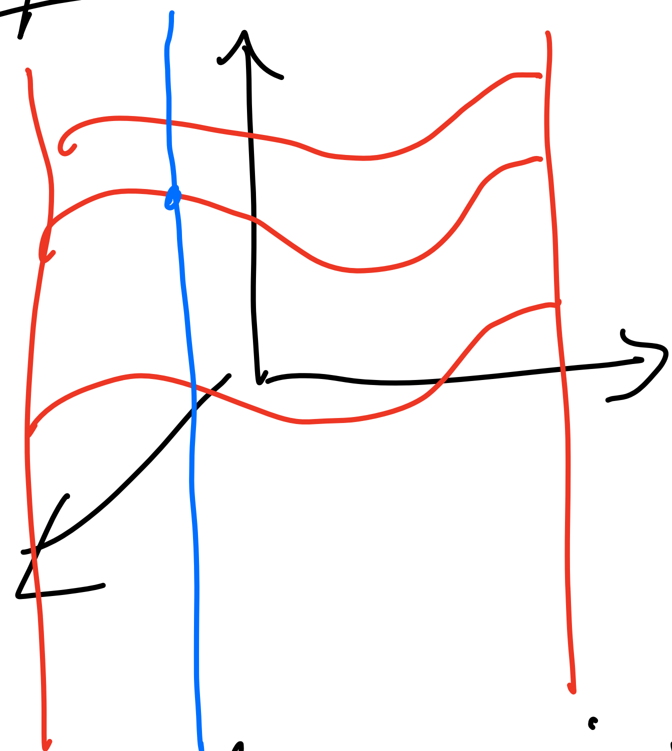
Example:

1) Trace of a plane is a line
2) Trace of a sphere



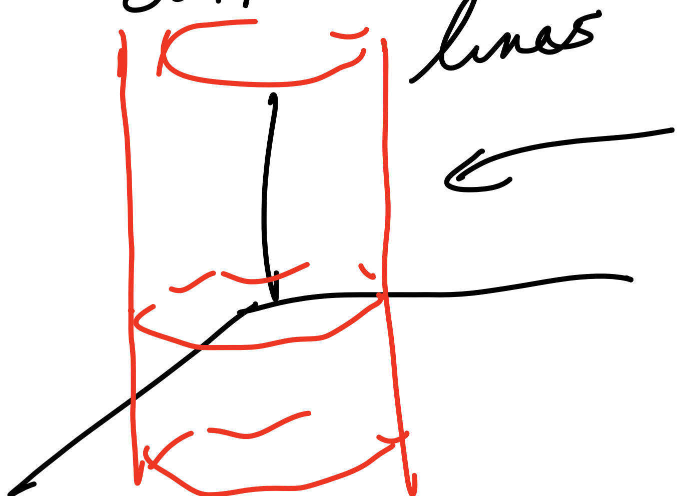
Usually we use only horizontal or vertical planes parallel to xy-plane, xz-plane, or yz-plane

Cylinders



Start with a curve in xy-plane and move it vertically

Surface is a union of vertical lines



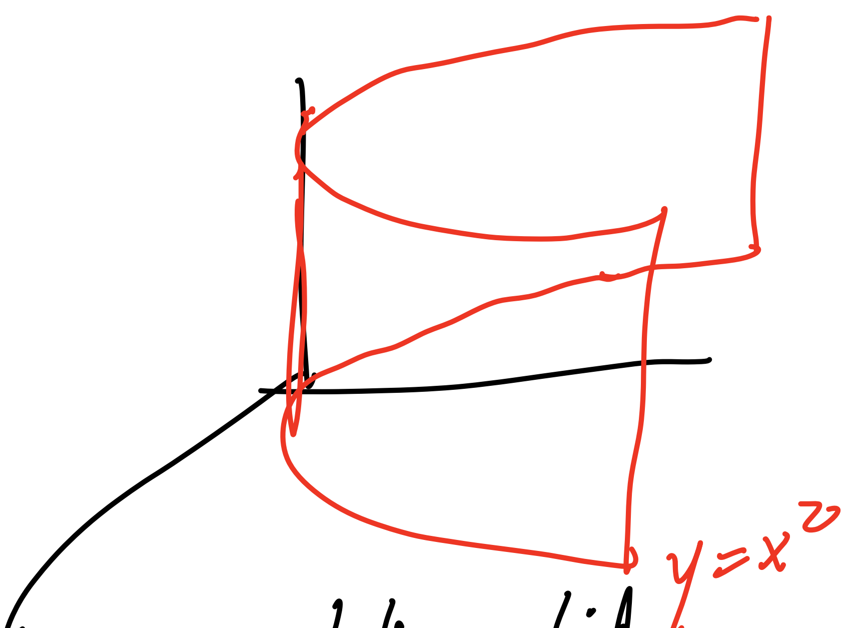
Equation of a cylinder

$f(x,y) = c$ (no z in equation)

So if $(x, y, 3)$ is a solution,

so is $(x, y, -1011)$

z can be anything



parabolic cylinder

Every horizontal trace is same curve

Circular Paraboloid

Start with a vertical parabola
(symmetric around z-axis)
and rotate it around z-axis

Horizontal traces
1) Starts empty
2) At height 0, single point
3) Height >0 ⇒ circles of increasing radii as height increases

Vertical trace of planes through z-axis is a parabola (always the same one)

<u>Equation</u>

$$z = \frac{x^2+y^2}{a^2}$$

Equation of a horizontal slice

Horizontal plane at height h is $z=h$

Intersection of paraboloid with this plane is

$$z = \frac{x^2+y^2}{a^2} \text{ and } z=h$$

$$\Rightarrow \frac{x^2+y^2}{a^2} = h \Rightarrow x^2+y^2 = ha^2$$

$\Rightarrow h<0 \Rightarrow$ Nothing

$\Rightarrow h=0 \Rightarrow x^2+y^2=0 \Rightarrow (0,0)$

$\Rightarrow h>0 \Rightarrow$ circle of radius

$r=\sqrt{h a^2}=a\sqrt{h}$

Elliptic paraboloid

$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

Horizontal traces are rescaled ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=h$$

$h>0$:

$$\frac{x^2}{(a\sqrt{h})^2}+\frac{y^2}{(b\sqrt{h})^2}=1$$

$$\frac{x^2}{(\;)^2}+\frac{y^2}{(\;)^2}=1 \quad \text{Ellipse}$$

Hyperbolic Paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2} \quad \left(z = \frac{-x^2}{a^2} + \frac{y^2}{b^2} \right)$$

1) Horizontal trace
$z = h$

$h > 0$ $\quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = h$

$$\frac{x^2}{(a\sqrt{h})^2} - \frac{y^2}{(b\sqrt{h})} = 1$$

Same for all b

$\Rightarrow$ hyperbola with asymptotes

$$y = \frac{b\sqrt{h}}{a\sqrt{h}} = \frac{b}{a}$$

passing through $(\pm a\sqrt{h}, 0)$

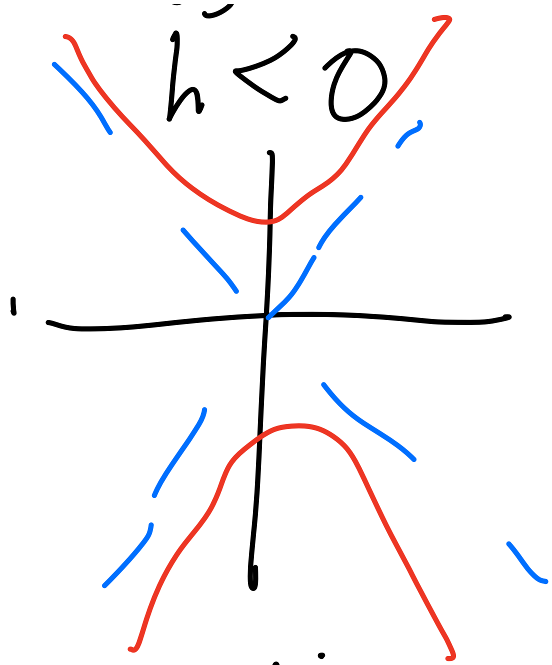
As h increases, hyperolas more apart

$h < 0$ $\quad \frac{x^2}{(a\sqrt{h})^2} - \frac{y^2}{(b\sqrt{h})^2} = -1 \Rightarrow$ hyperbolas switch regions

$(0, \pm b\sqrt{h}$

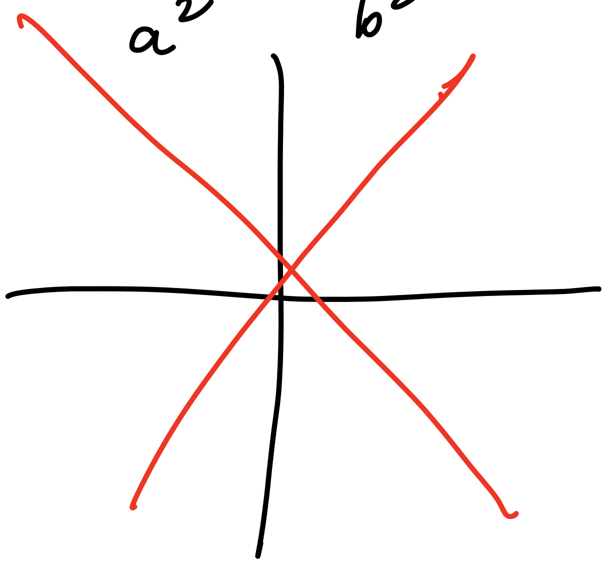
$h > 0$

$h < 0$



$h = 0$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \Rightarrow$$ 2 lines crossing

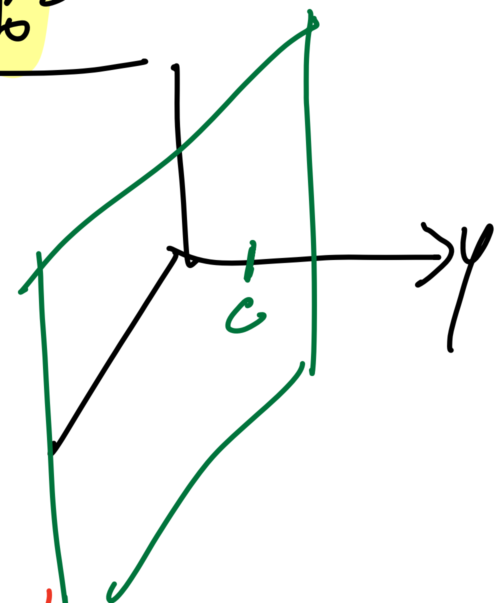


Vertical traces of

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

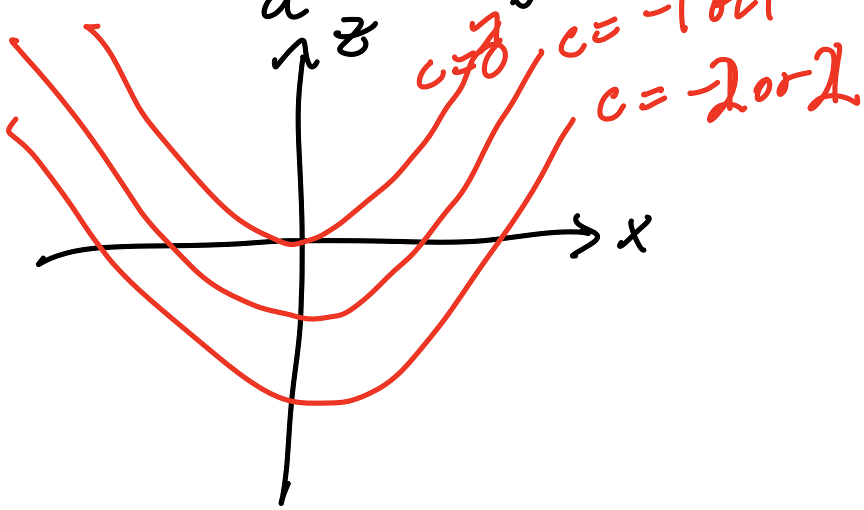
Parallel to xz-plane

$y = c$



Plug into equation

$$z = \frac{x^2}{a^2} - \frac{c^2}{b^2}$$

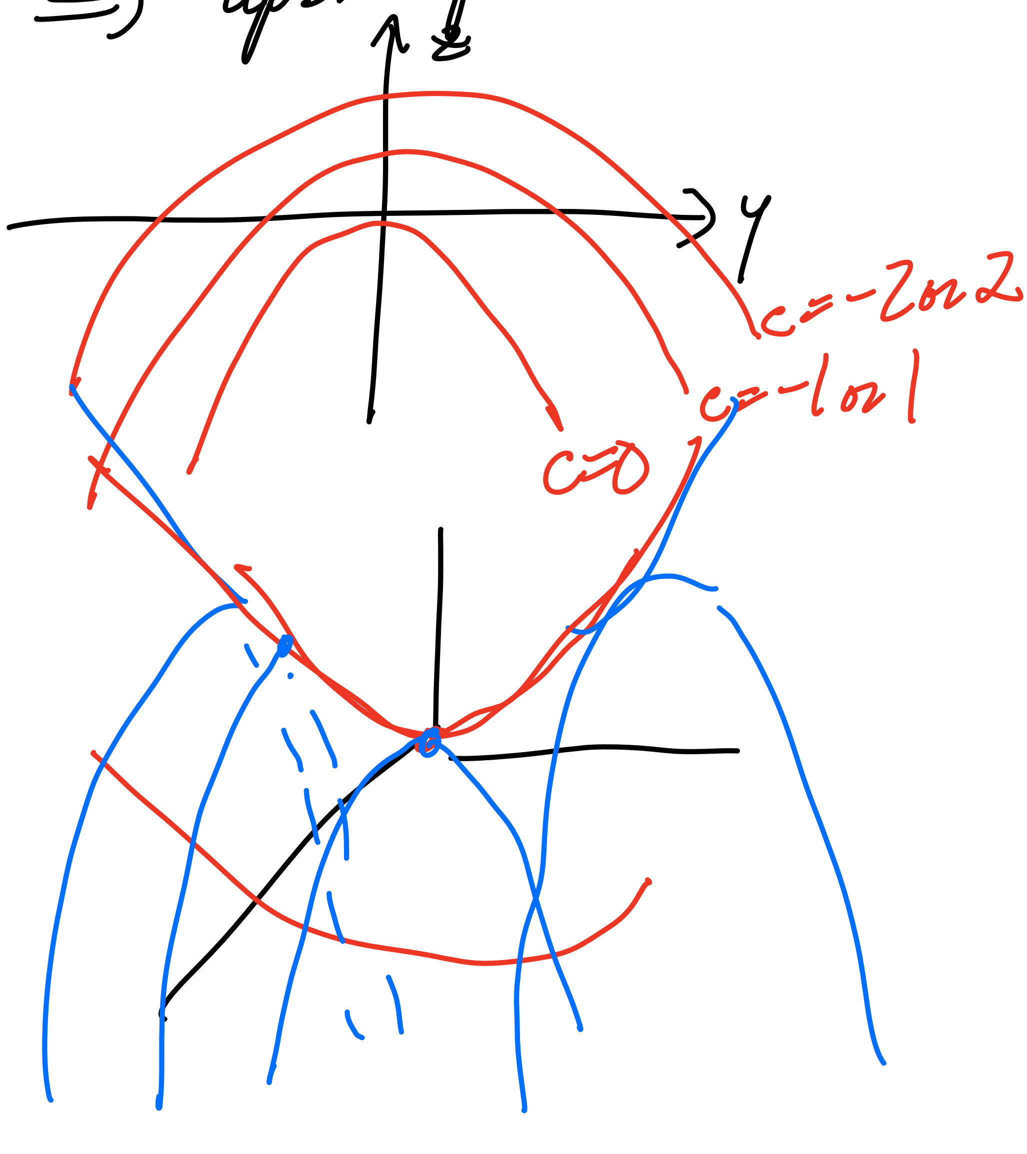


Parallel to yz-plane

$x = c$

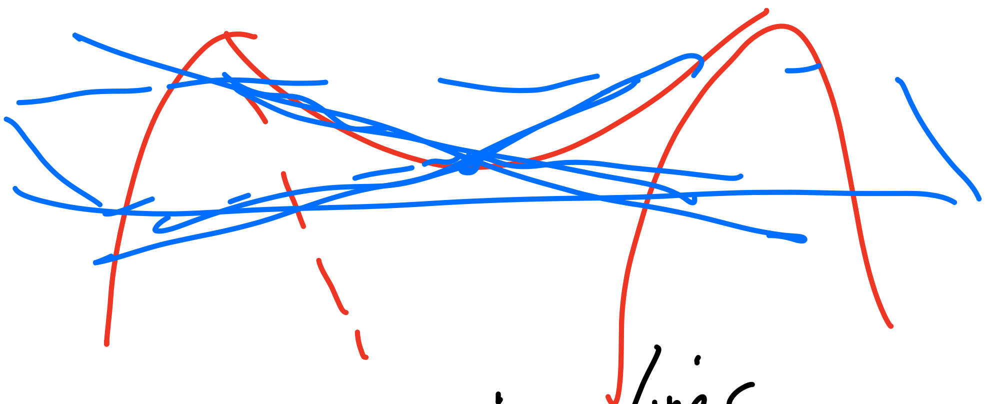
$\Rightarrow \quad z = \frac{-y^2}{b^2} + \frac{c^2}{a^2}$

$\Rightarrow$ upside parabolas



Saddle surface

Intersection with $z=0$



is 2 crossing lines

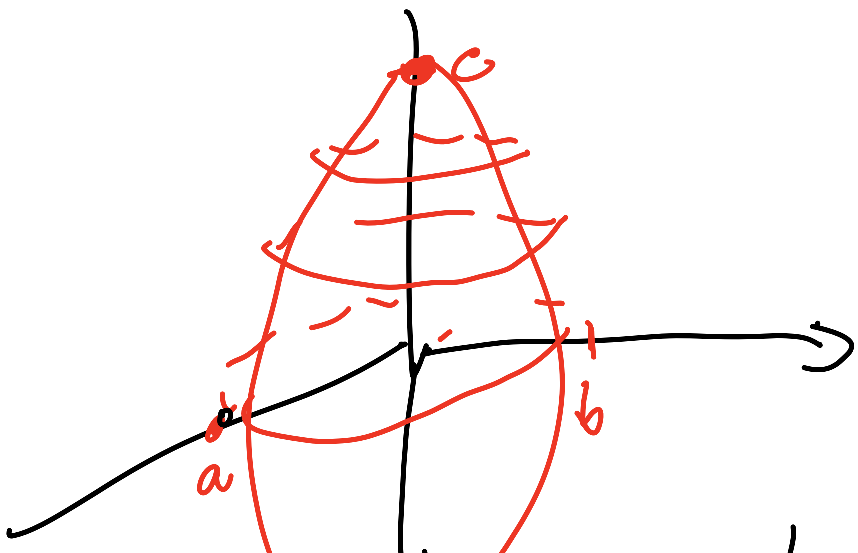
Hyperbolic paraboloid

Horizontal traces are hyperbolas

Vertical traces parallel to xz or yz plane are parabolas

Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



Horizontal trace $z=h$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{h^2}{c^2}$$

$\Rightarrow$ $h>c$ or $h<-c \Rightarrow$ empty

$h=c$ or $h=-c \Rightarrow x=y=0$

$h=0 \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \pm 1$$

Assume $a = b = 1$

$$x^2 + y^2 - z^2 = 1$$

$$x^2 + y^2 = 1 + z^2$$

Horizontal traces: $z = h$

$$x^2 + y^2 = 1 + h^2 \geq 0$$

$\Longrightarrow$ circles that increase in radius as $|h|$ increases.

At $h = 0$, circle of radius 1

1-sheeted hyperboloid

$$x^2+y^2-z^2=1$$

$$x^2+y^2=1+z^2$$

Another type of hyperboloid $x^2+y^2-z^2=-1$

$x^2+y^2=-1+z^2$

Horizontal traces: $z=h$

$x^2+y^2=-1+h^2$

If $-1<h<1$: empty

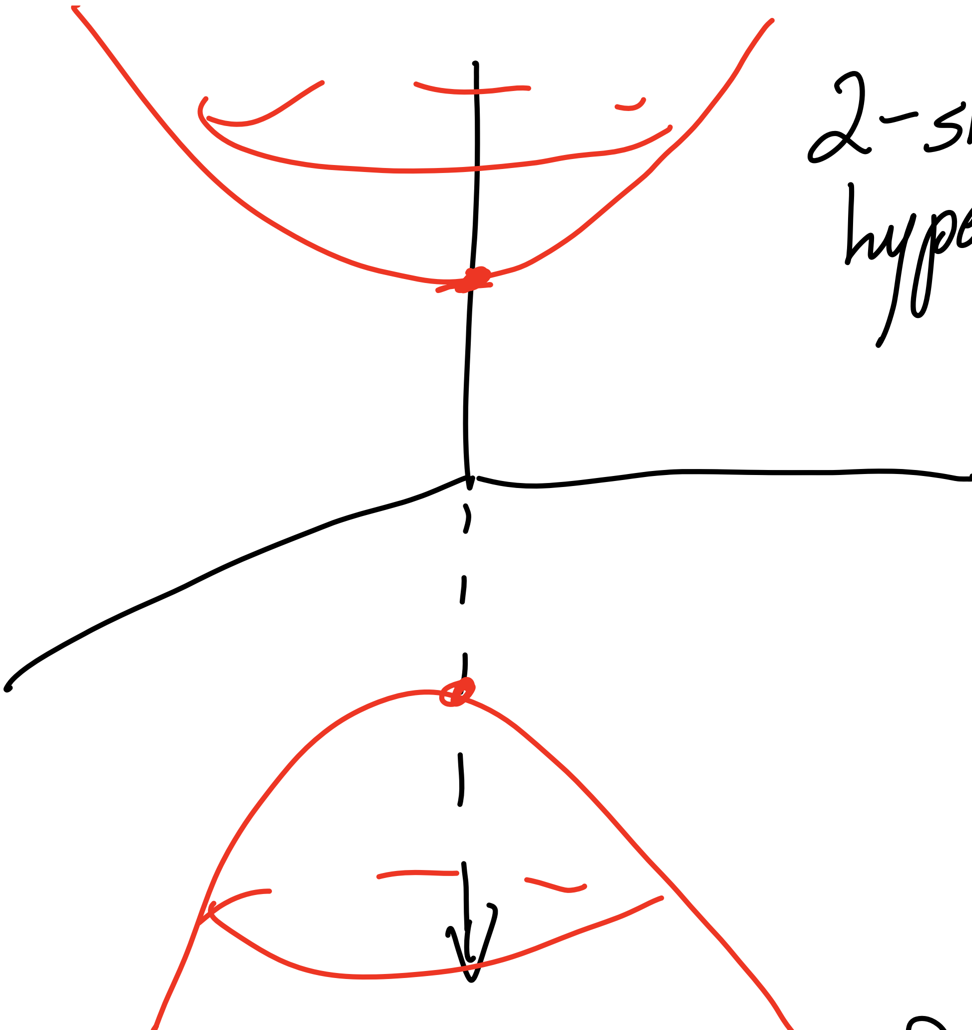
$h=1$ or $-1 \Rightarrow x^2+y^2=0$

$\Rightarrow$ single point

$h>1$ or $h<-1$

$\Rightarrow$ get circle of radius $\sqrt{h^2-1}$

radius increases as $|h|$ increases



2-sheeted hyperboloid

$$\left.\begin{aligned} \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}&=1 \\ \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}&=-1 \end{aligned}\right\} \text{hyperboloids}$$

To figure out which is 1-sheeted and which is 2-sheeted,

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{z^2}{c^2}$

(b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1 + \frac{z^2}{c^2}$

Look at trace $z=0$

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ← ellipse

(b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$ ← empty

So (a) 1-sheeted

(b) 2-sheeted