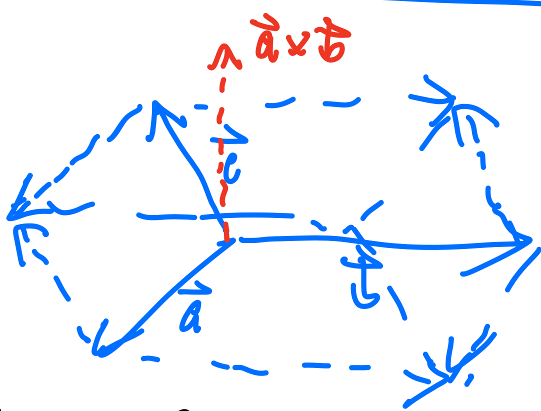
Volume of parallelopiped
parallelotope

tilted box



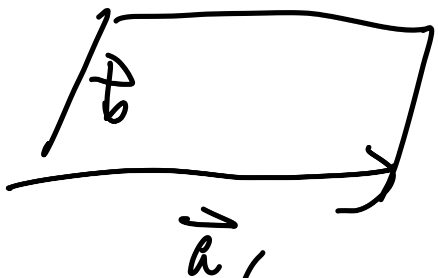
Volume of parallelopiped

= (area of base)(height)

height = scalar projection of $\vec{c}$ onto $\vec{a}\times\vec{b}$

So if $\vec{u} = \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$, then height $= \vec{c}\cdot\vec{u}$

Area of base $= |\vec{a} \times \vec{b}|$



$$\text{Volume} = (\text{area of base})(\text{height})$$

$$= |\vec{a} \times \vec{b}| \left| (\vec{c} \cdot \vec{u}) \right|$$

$$= |\vec{a} \times \vec{b}| \left| \left(\vec{c} \cdot \left(\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} \right) \right) \right|$$

$$= |\vec{a} \times \vec{b}| \left(\frac{1}{|\vec{a} \times \vec{b}|} \right) \left| \left(\vec{c} \cdot (\vec{a} \times \vec{b}) \right) \right|$$

$$= \left| \vec{c} \cdot (\vec{a} \times \vec{b}) \right|$$

Oriented volume

$(\vec{a}, \vec{b}, \vec{c})$ has positive orientation

$\Longrightarrow (\vec{a} \times \vec{b}) \cdot \vec{c} > 0$

Oriented volume

$$V = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$= \begin{cases} \text{volume if } (\vec{a}, \vec{b}, \vec{c}) \text{ has pos. or} \\ -\text{volume if neg orient.} \end{cases}$$

To check orientation of $(\vec{a}, \vec{b}, \vec{c})$

1) Draw them, check righthand rule

2) Calculate $(\vec{a} \times \vec{b}) \cdot \vec{c}$

Sign of this gives orientation

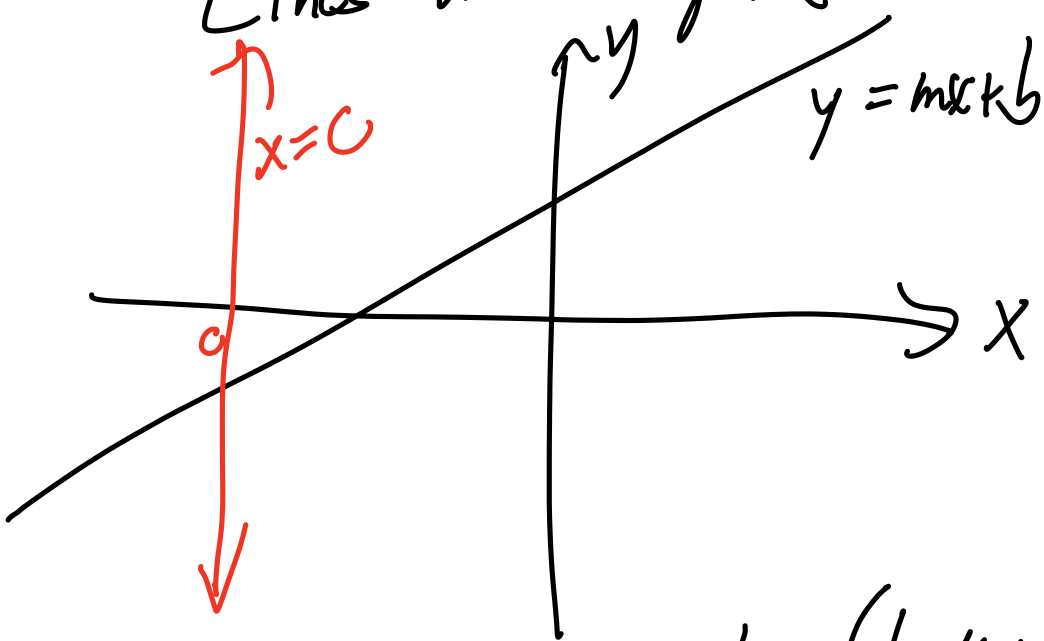
$(\vec{a}, \vec{b}, \vec{c})$ $+$

$\to (\vec{b}, \vec{a}, \vec{c})$ $-$ $\qquad (\vec{b}, \vec{c}, \vec{a})$ $+$

$(\vec{a}, \vec{c}, \vec{b})$ $-$ $\qquad (\vec{c}, \vec{a}, \vec{b})$ $+$

$\cancel{(\vec{a}, \vec{b}, \vec{c})}$

Lines in 2-space



Graph: $y = mx+b$ (but no vertical lines)

Vertical line: $x = c$

Solution to linear equation

$$Ax + By = C$$

$$2x - 3y = 1 \qquad 2x = 1$$

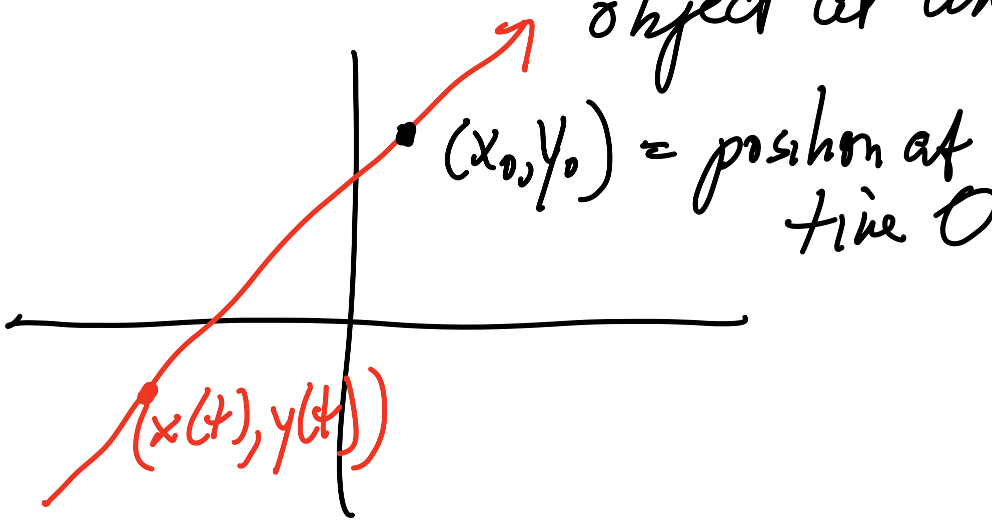
$$y = \frac{2}{3}x - \frac{1}{3} \qquad x = \frac{1}{2}$$

Parameterization

View line as the path of an object

t = time

$(x(t), y(t))$ = location of object at time t

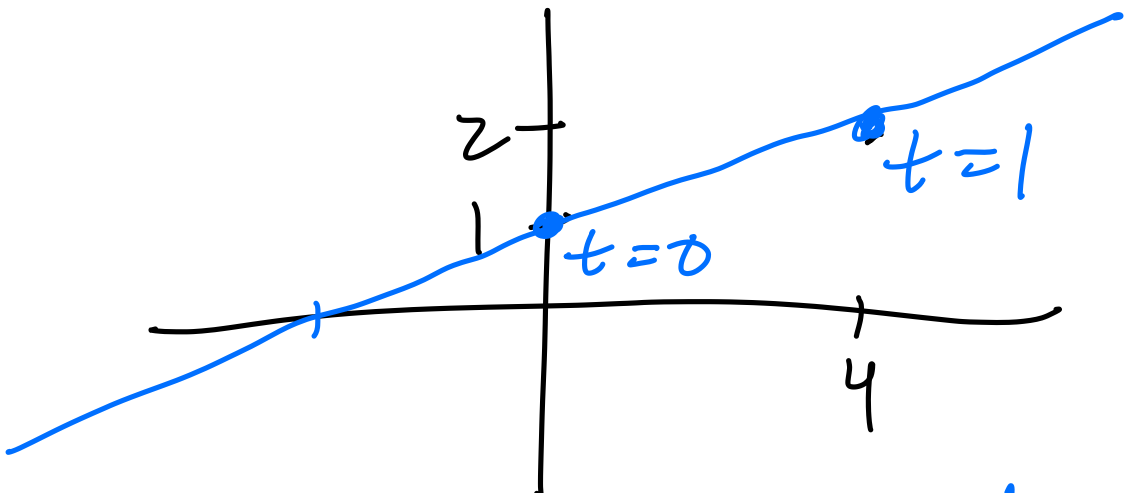


$$x(t) = x_0 + at$$
$$y(t) = y_0 + bt$$
$$(x(0), y(0)) = (x_0, y_0)$$

P_0

Parameterization of line is writing x, y as linear functions of a new variable t

t called "parameter"



$y = +\frac{1}{4}x + 1$ graph

$-x + 4y = 4$ linear eq'n

$(x,y) = (0,1) + t\langle 4,1 \rangle$ parameterization

(Many possible parameterizations)

$$x(t) = 4t$$
$$y(t) = 1 + t$$

2 distinct points $\Longrightarrow$ unique line

$(3,4)$, $(-1,17)$

$\Longrightarrow$ find equation of line

Parameterization

$$(x,y)(0) = (3,4)$$
$$(x,y)(1) = (-1,17)$$

$$(x,y)(t) = (3,4) + t\big((-1,17) - (3,4)\big)$$
$$(x,y)(1) = (3,4) + \big((-1,17) - (3,4)\big)$$
$$= (-1,17)$$

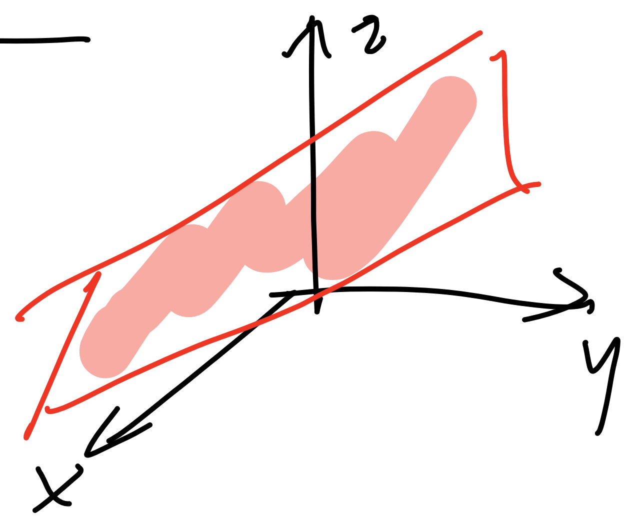
~~Lines~~ in 3-space

Planes

- Graph

$z = ax + by + c$

(but no vertical planes)



- Solution to linear equation

$Ax + By + Cz = D$

- Parameterization

Need 2 parameters because plane is 2-dimensional

$(x, y, z)(s, t)$

$= (x_0, y_0, z_0) + s\langle v_1, v_2, v_3\rangle$

$+ t\langle w_1, w_2, w_3\rangle$

$$x(s,t) = x_0 + s v_1 + t w_1$$
$$y(s,t) = y_0 + s v_2 + t w_2$$
$$z(s,t) = z_0 + s v_3 + t w_3$$

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3 non-collinear points
determine a unique plane

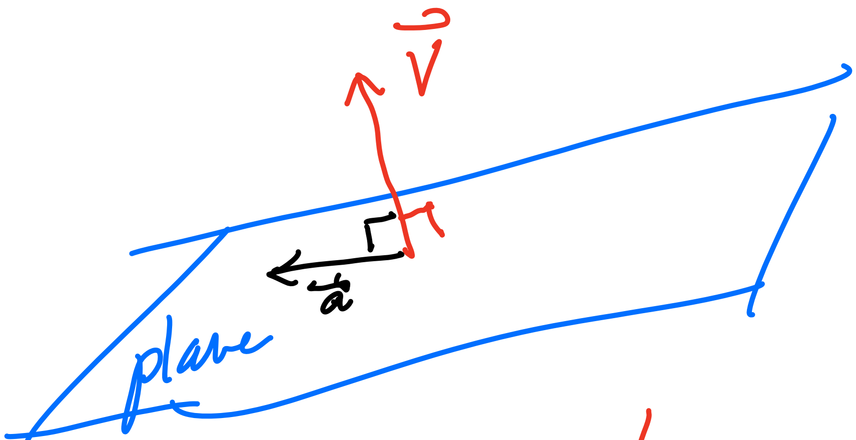
$(1,1,0)$, $(0,0,-2)$, $(0,3,0)$

- Solve for graph or linear equation of plane
- Parameterization

$$\boxed{\begin{aligned} (x,y,z)(0,0) &= (1,1,0) \\ (x,y,z)(1,0) &= (0,0,-2) \\ (x,y,z)(0,1) &= (0,3,0) \end{aligned}}$$

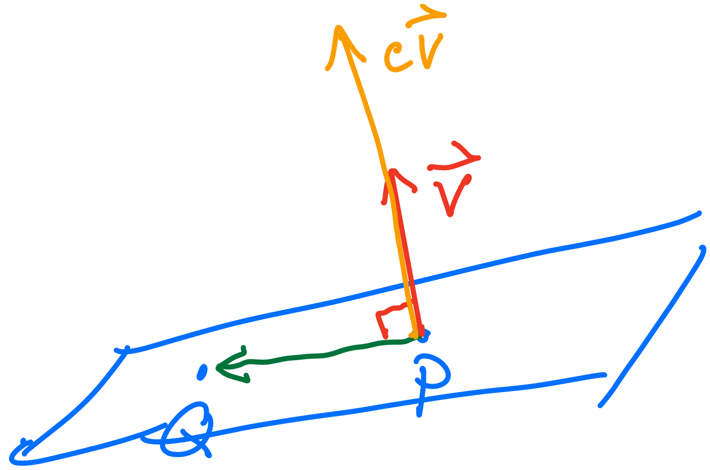
$$\boxed{\begin{aligned}(x,y,z)(s,t) = {} & (1,1,0) \\ & + s\big((0,0,-2)-(1,1,0)\big) \\ & + t\big((0,3,0)-(1,1,0)\big)\end{aligned}}$$

Normal vector plane



$\vec{v}$ is normal to a plane if it makes a right angle with plane

Same as: Any vector parallel to plane is orthogonal to $\vec{v}$



P, Q in plane

$Q - P$ vector parallel to plane

$\vec{v}$ normal vector of plane

nonzero

$$\Longrightarrow (Q-P) \perp \vec{v}$$

orthogonal

$$\Longrightarrow \vec{v} \cdot (Q-P) = 0$$

So if P in plane, $\vec{v}$ parallel,
then

Q in plane $\Longleftrightarrow (Q-P)\cdot\vec{v} = 0$

A plane uniquely determined by one point in the plane and a nonzero normal vector

$P = (x_0, y_0, z_0)$

$\vec{v} = \langle a, b, c \rangle$

$Q = (x, y, z)$ lies in plane

$\Updownarrow$

$(Q - P) \cdot \vec{v} = 0$

$\Updownarrow$

$\big((x, y, z) - (x_0, y_0, z_0)\big) \cdot \langle a, b, c \rangle = 0$

$\Updownarrow$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$ax + by + cz = ax_0 + by_0 + cz_0$

$$ax + by + cz = d$$

a, b, c constant

d constant

Compare:

$\vec{v} = \langle a, b, c \rangle$

a, b, c coefficients of equation of linear

Components of a normal vector for a plane $\iff$ Coefficients of x, y, z of a linear equation for the plane

$$2x + 3y - 17z = 23$$

Find a normal vector

Answer: $\langle 2, 3, -17 \rangle$

Linear equation $\longleftrightarrow$ point and normal vector

$$2x+3y-17z=23 \Longrightarrow \vec{v}=\langle 2,3,-17\rangle$$

$$P=\left(\frac{23}{2},0,0\right)$$

$$x+y-z=-1 \Longleftarrow \vec{v}=\langle 1,1,-1\rangle$$

$$0+2-3=-1 \qquad P=(0,2,3)$$

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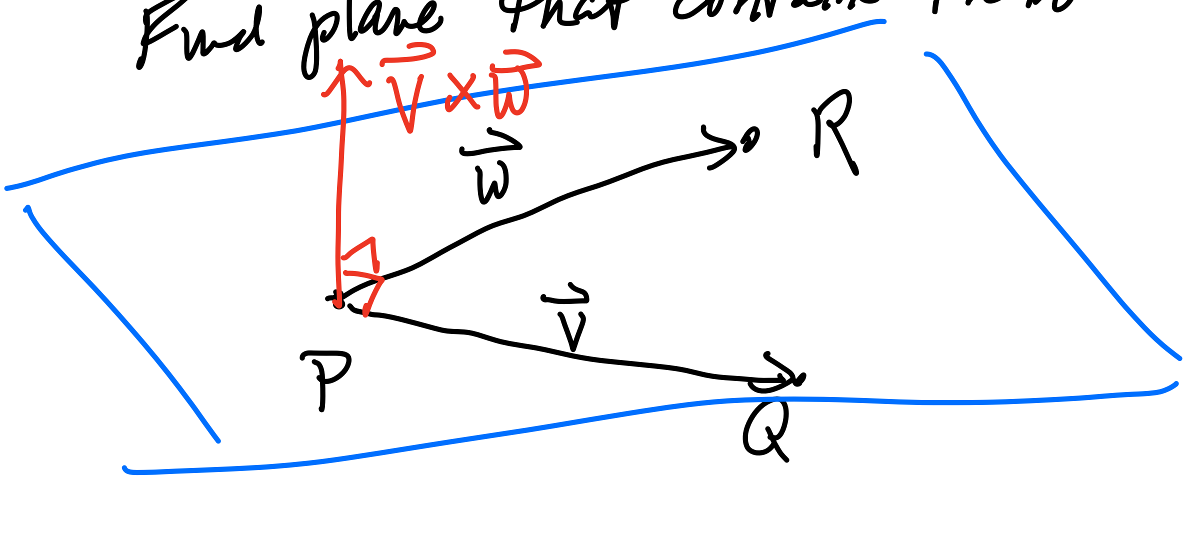
3 points in space

Find plane that contains them



P, Q, R not collinear

$\Rightarrow \vec{v} = Q - P,\ \vec{w} = R - P$

are not parallel

$\Leftrightarrow \vec{v} \times \vec{w} \neq 0$

Since $\vec{v} \times \vec{w} \perp$ both $\vec{v}$ and $\vec{w}$

it's normal to plane

Example

$A = (1,1,1),\ B = (-2,0,1),\ C = (1,0,0)$

Find plane containing A, B, C

$$\left.\begin{aligned} \vec{v} &= A - C = \langle 0, 1, 1 \rangle \\ \vec{w} &= C - B = \langle 3, 0, -1 \rangle \end{aligned}\right\} \text{Parallel to plane}$$

$$\vec{n} = \vec{v} \times \vec{w}$$

$$= (\vec{j} + \vec{k}) \times (3\vec{i} - \vec{k})$$

$$= (-1 \qquad)\vec{i}$$

$$+ (+3 \qquad)\vec{j}$$

$$+ (-3 \qquad)\vec{k}$$

$$= -\vec{i} + 3\vec{j} - 3\vec{k} = \langle -1, 3, -3 \rangle$$

$$\boxed{\begin{aligned} \vec{n} &= \langle -1, 3, -3 \rangle \\ P &= (1, 0, 0) \end{aligned}}$$

normal vector

point

$$(-1)x + 3y + (-3)z = -1$$

$$(-1)1 + 3(0) + (-3)0 = -1$$

Find 3 points (3 solutions)

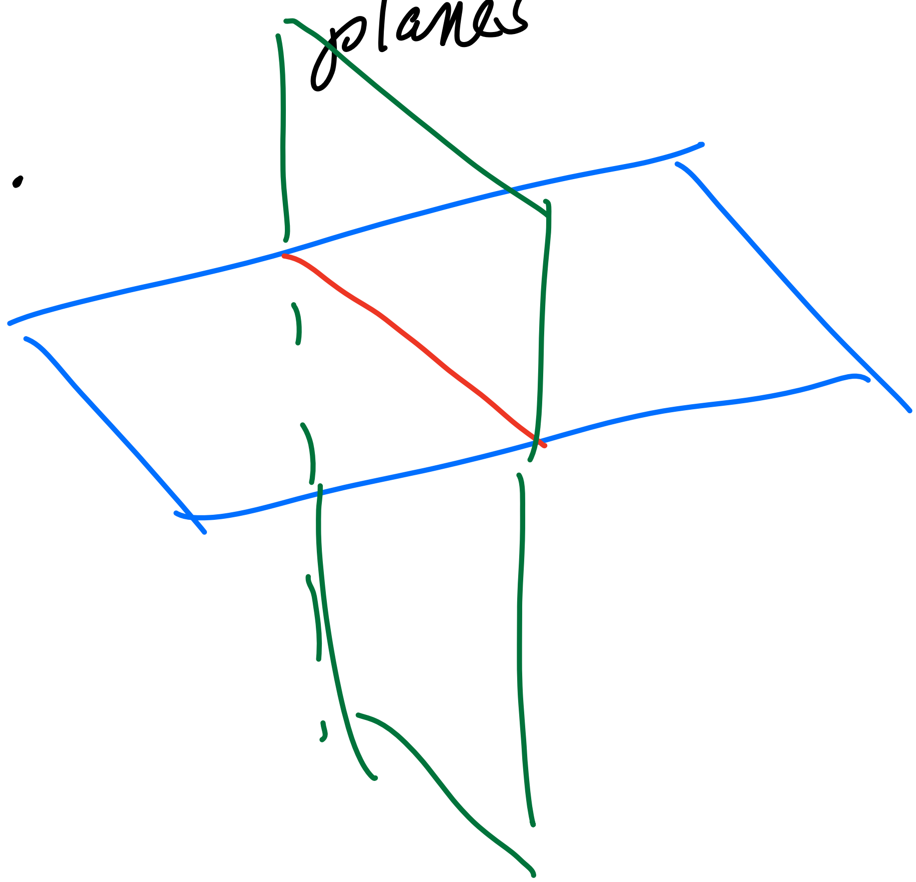
$y = z = 0 \Rightarrow -x = -1 \Rightarrow (1, 0, 0)$

$x = z = 0 \Rightarrow 3y = -1 \Rightarrow (0, -\frac{1}{3}, 0)$

$x = y = 0 \Rightarrow -3z = -1 \Rightarrow (0, 0, \frac{1}{3})$

3 non-collinear points

Line in 3-space

- Intersection of 2 non-parallel planes
- 

- 2 points uniquly determine line
- Intersection of 2 planes

$\Downarrow$

Solutions to a system of 2 linear equations with nonparallel coefficents (normal vector)

Example
$$\boxed{\begin{aligned} x + y \quad &= 4 \\ y - z &= 1 \end{aligned}}$$

- Parameterization of line

$$(x,y,z)(t) = (x_0, y_0, z_0) + t\langle v_1, v_2, v_3\rangle$$

Line through 2 points

$P = (1,1,0)\,,\ (0,0,2) = Q$

Parameterization

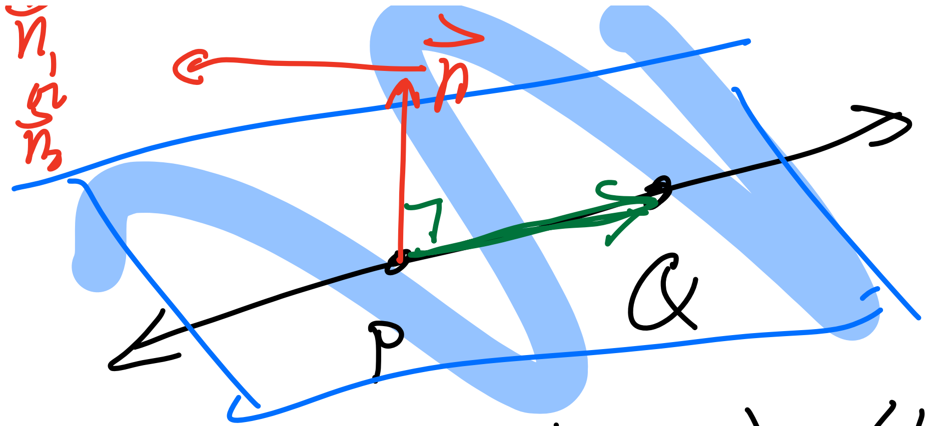
$$(x,y,z)(t) = (1,1,0) + t\big((0,0,2) - (1,1,0)\big)$$
$$= (1,1,0) + t\langle -1,-1,2\rangle$$
$$= (1-t,\ 1-t,\ 2t)$$

Intersection of 2 planes

Look for 2 nonparallel planes containing lines.

$\Rightarrow$ look for 2 non-parallel nonzero normal vectors $\vec{n}_1, \vec{n}_2$

$\Rightarrow$



$$Q - P = (0,0,2) - (1,1,0)$$

$$= \langle -1, -1, 2 \rangle$$

$$\left.\begin{aligned} \vec{n}_1 &= \langle 1, -1, 0 \rangle \\ \vec{n}_2 &= \langle 0, 2, 1 \rangle \end{aligned}\right\} \text{Both normal to } Q - P$$

both contain P and Q

Plane with normal $\vec{n}_1$ and point P contains line

Plane with normal $\vec{n}_2$ and point P also contains line

Planes are not parallel

So intersection of these
2 planes is the line
through P, Q

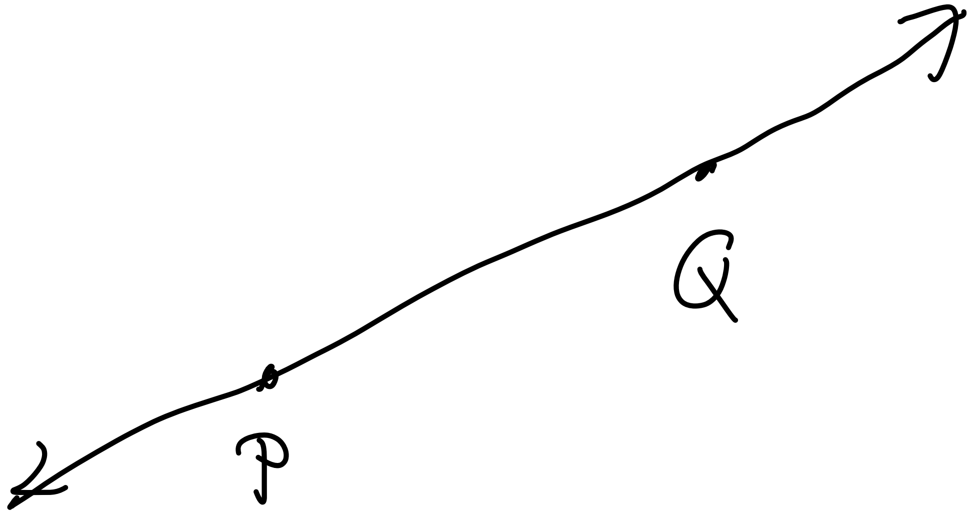
$\Downarrow$

Can write equations for
the 2 planes

$\Downarrow$

get 2 linear equations
whose solutions form the line

What we did:
Start with points P, Q
and found 2 planes
whose intersection is the line
through P and Q



- Parameterization
- Intersection of 2 planes

<u>Preview</u>

<u>Quadric surface</u>

Solution to a quadratic polynomial

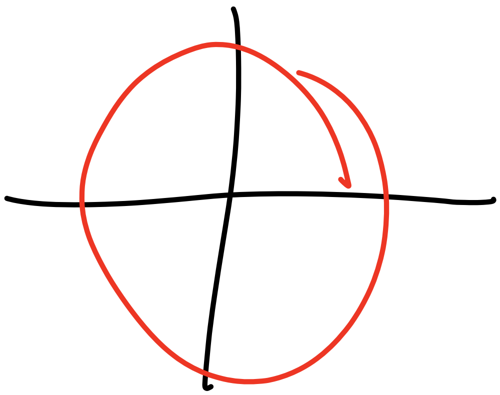
$$ax^2+bxy+cy^2+dxz+eyz+fz^2=g$$

$$(ax^2+bxy+cy^2=d) \leftarrow \text{quadric curve}$$

Examples

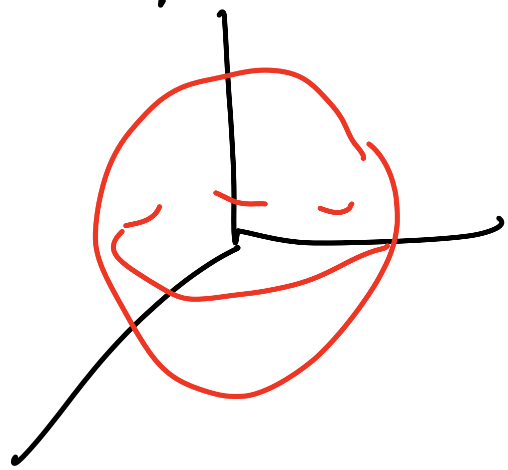
2D

$x^2+y^2=r^2$

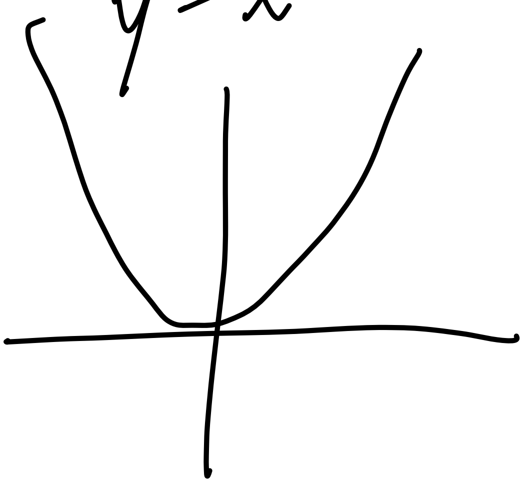


3D

$x^2+y^2+z^2=r^2$

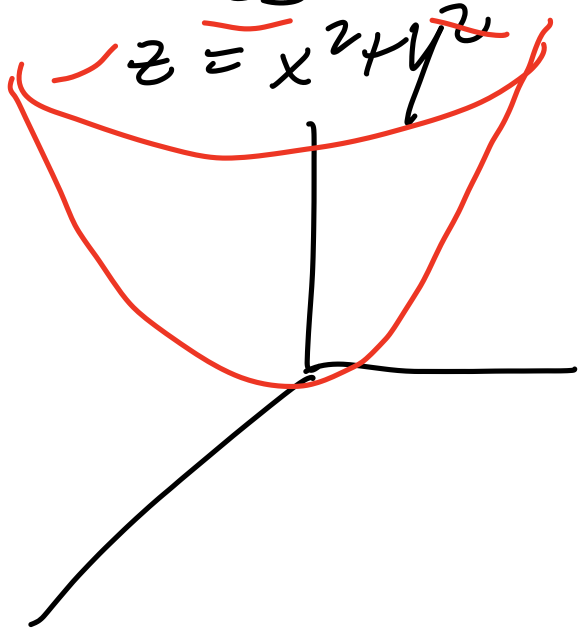


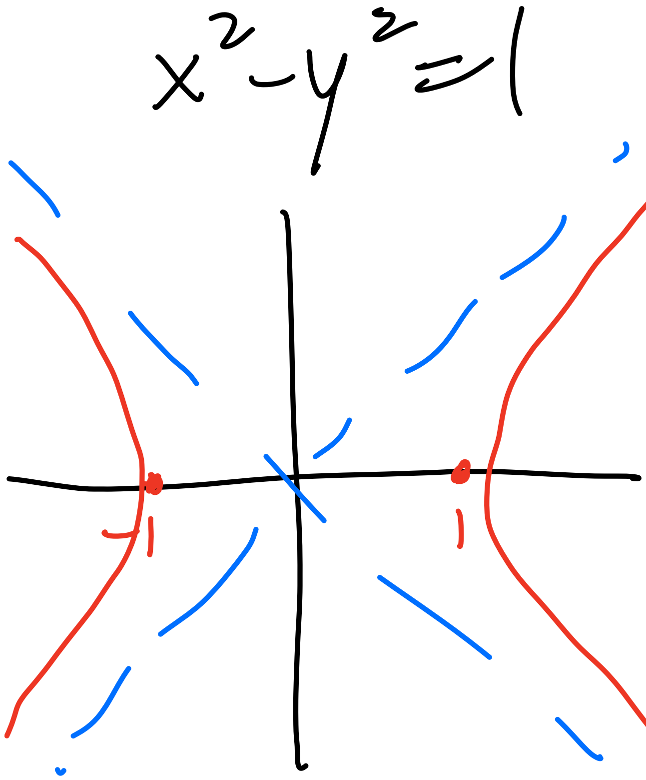
$y=x^2$

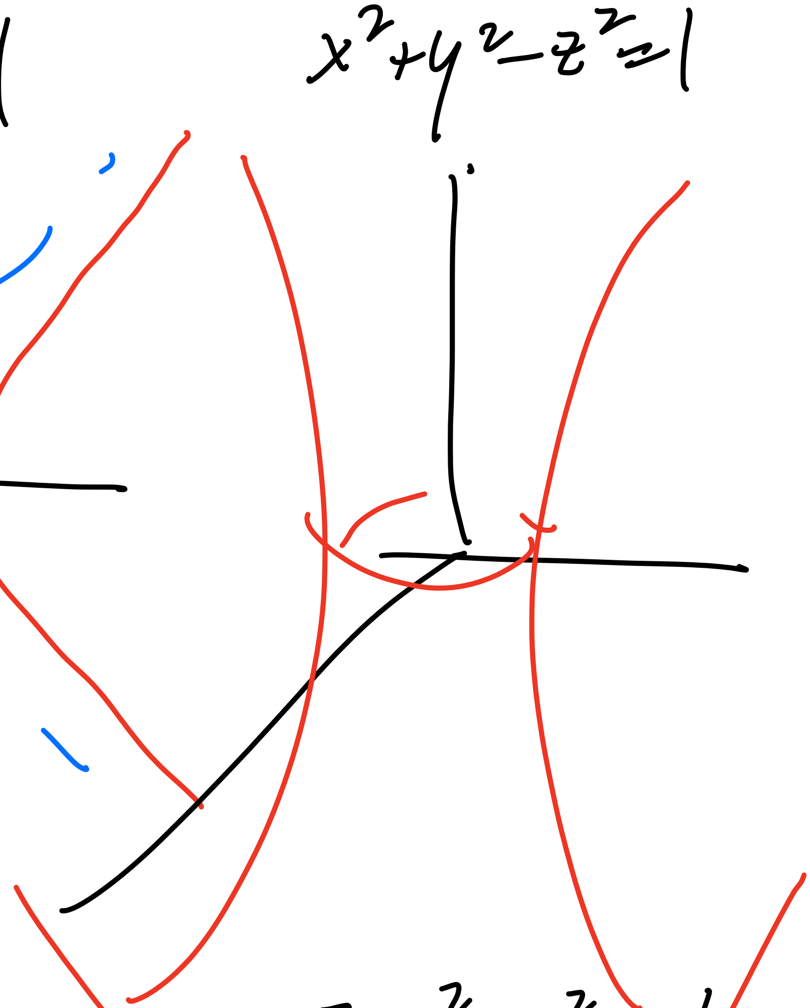


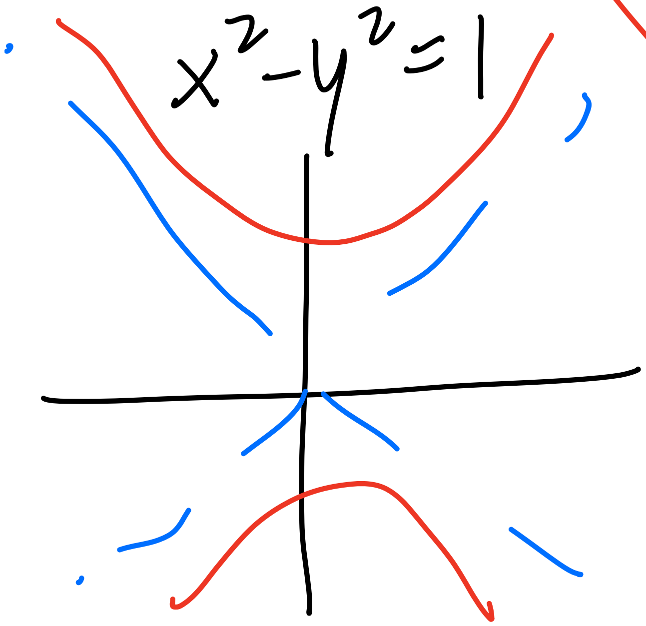
3D

$z=x^2+y^2$




$x^2 - y^2 = 1$


$x^2 + y^2 - z^2 = 1$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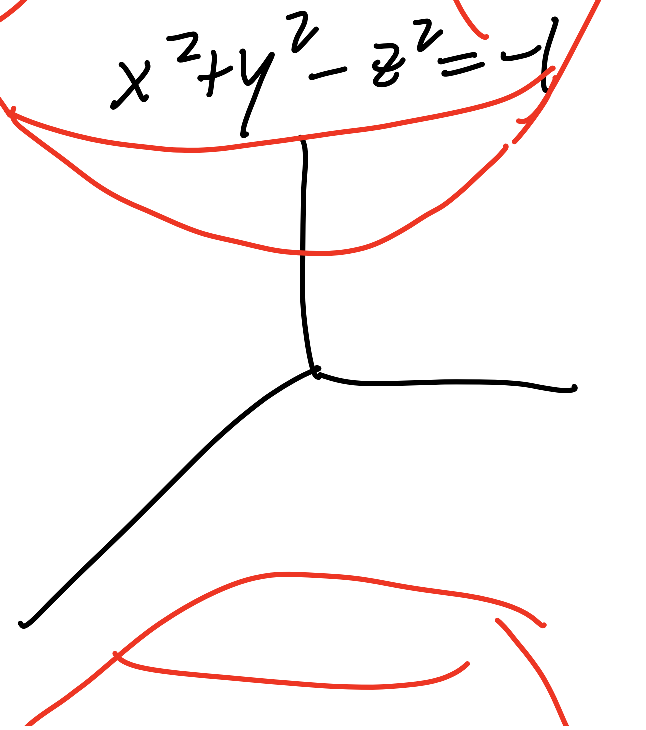
$x^2 + y^2 - z^2 = -1$