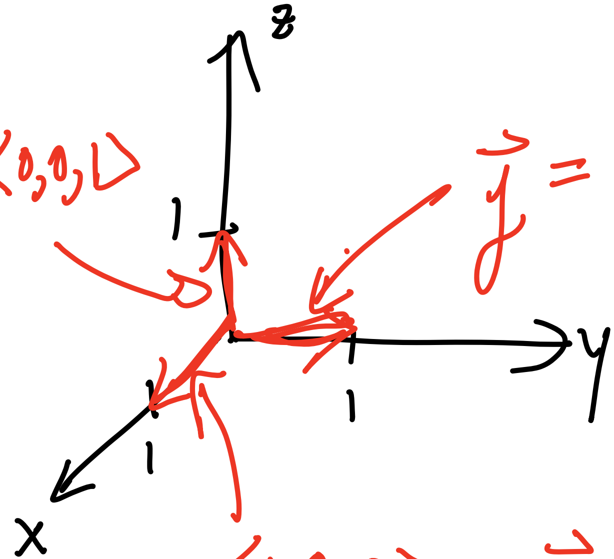
Standard unit vectors



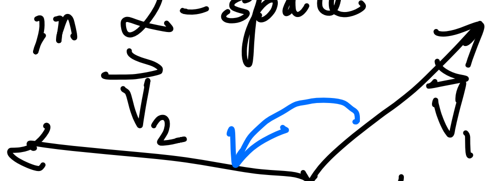
$\vec{k} = \langle 0,0,1 \rangle = \vec{e}_3$

$\vec{j} = \langle 0,1,0 \rangle = \vec{e}_2$

$\langle 1,0,0 \rangle = \vec{\imath} = \hat{\imath} = \mathbf{i}$ (bold)

$\vec{\imath} = \vec{e}_1$

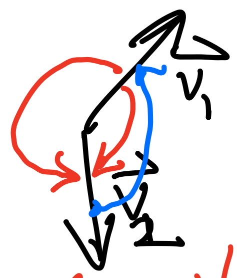
Orientation in 2-space

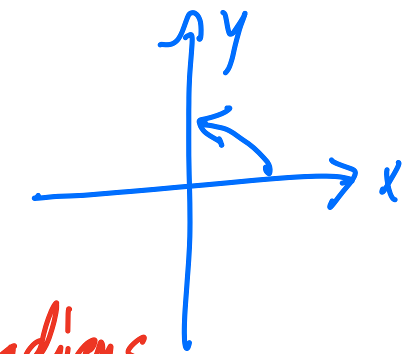


Counterclockwise means from positive x-axis toward positive y-axis

$\vec{v}_2$ lies less than π radians counterclockwise of $\vec{v}_1$

$\Rightarrow$ $(\vec{v}_1, \vec{v}_2)$ has positive orientation





$\vec{v}_2$ lies less than π radians clockwise of $\vec{v}_1$

$\Longrightarrow$ $(\vec{v}_1, \vec{v}_2)$ has negative orientation.

$\vec{v}_1$ lies less than π radians counterclockwise of $\vec{v}_2$ $\Longrightarrow$ $(\vec{v}_2, \vec{v}_1)$ has positive orientation

---

Cross product in 2-space

$\vec{v}, \vec{w}$ vectors in 2-space

If $(\vec{v}, \vec{w})$ has positive orientation, then $\vec{v} \times \vec{w}$ = area of parallelogram with sides $\vec{v}, \vec{w}$

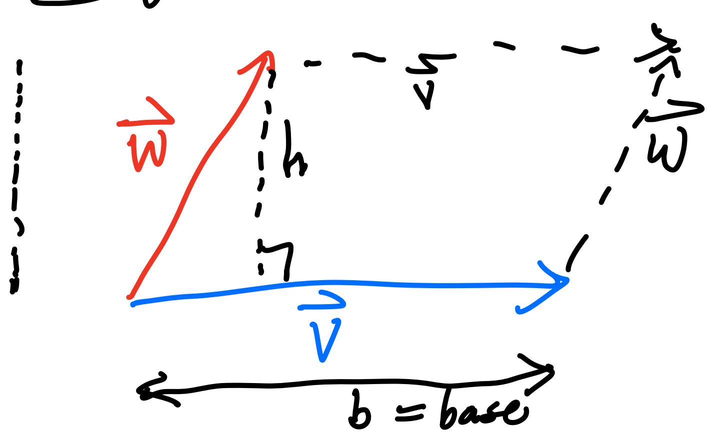
If $(\vec{v}, \vec{w})$ has negative orientation,
$\vec{v} \times \vec{w} = -(\text{area of parallelogram})$

$\vec{v} \times \vec{w}$ = oriented area of parallelogram

$\vec{v}, \vec{w}$ vectors
$\vec{v} \times \vec{w}$ scalar

$$\vec{w} \times \vec{v} = -(\text{area})$$
$$= -\vec{v} \times \vec{w}$$

Cross product is not commutative
It is anticommutative



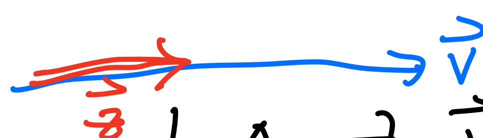
$$\vec{v} \times \vec{w} = bh \text{ if } (\vec{v}, \vec{w}) \text{ has positive orientation}$$

$$= -bh \text{ if } (\vec{v}, \vec{w}) \text{ has negative orientation}$$

$$\vec{v} \times \vec{v} = -(\vec{v} \times \vec{v})$$

$$2(\vec{v} \times \vec{v}) = 0$$

$$\vec{v} \times \vec{v} = 0 \text{ for any } \vec{v}$$



Here, $h = 0 \Rightarrow \vec{v} \times \vec{z} = \vec{z} \times \vec{v} = 0$

$h = 0$

$$\Rightarrow \vec{z} \times \vec{v} = \vec{v} \times \vec{z} = 0$$

$\vec{v}, \vec{w}$ orthogonal

$\Longrightarrow \quad b = |\vec{v}|$

$\quad\quad h = |\vec{w}|$

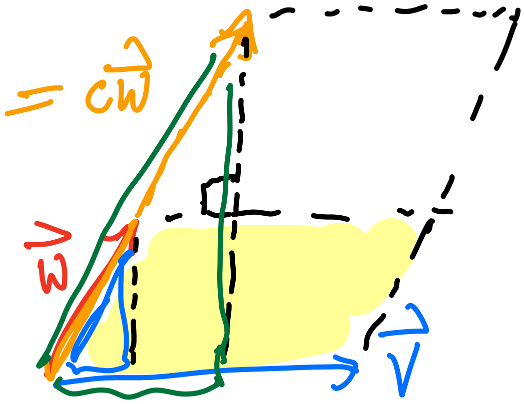
$$\vec{v} \times \vec{w} = \begin{cases} |\vec{v}||\vec{w}| & \text{if pos orient} \\ -|\vec{v}||\vec{w}| & \text{if neg orient} \end{cases}$$

$$\langle 2, 1 \rangle \times \langle -1, 2 \rangle$$

$$\left(\sqrt{4+1}\right)"\left(\sqrt{4+1}\right) = 5$$

Properties of cross product

$\frac{3}{2}\vec{w} = c\vec{w}$



$\vec{v} \times \frac{3}{2}\vec{w} =$

width $= |\vec{v}|$ stays the same

height $= \frac{3}{2} h_{(\vec{v}\times\vec{w})}$

$$\therefore \quad \vec{v} \times \left(\frac{3}{2}\vec{w}\right) = \frac{3}{2}\left(\vec{v}\times\vec{w}\right)$$

More generally

$$(c\vec{v}) \times \vec{w} = \vec{v} \times (c\vec{w}) = c(\vec{v}\times\vec{w})$$

$$(2\vec{v}) \times (2\vec{w}) \neq 2(\vec{v}\times\vec{w})$$

$$= 4(\vec{v}\times\vec{w})$$

$$\vec{w} = \vec{w}_1 \times \vec{w}_2$$

Yellow triangles have same area

Compare area of parallelogram with sides $\vec{v}, \vec{w}$

$$= \vec{v} \times \vec{w}$$

to sum of areas of parallelograms $(\vec{v}, \vec{w})$ and $(\vec{v}, \vec{w}_2)$

From picture,

$$\vec{v} \times \vec{w} = \vec{v} \times \vec{w}_1 + \vec{v} \times \vec{w}_2$$

$$\vec{v} \times (\vec{w}_1 + \vec{w}_2) = \vec{v} \times \vec{w}_1 + \vec{v} \times \vec{w}_2$$

$$(\vec{v}_1 + \vec{v}_2) \times \vec{w} = \vec{v}_1 \times \vec{w} + \vec{v}_2 \times \vec{w}$$

$\Rightarrow$ Cross product is distributive with respect to addition

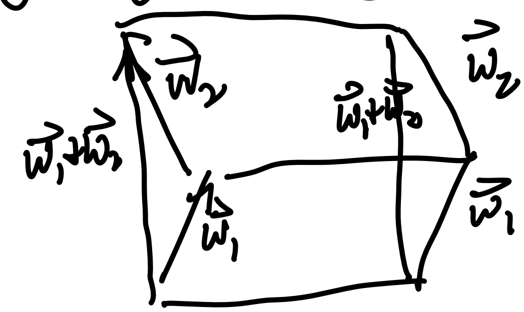
Cross product is not associative

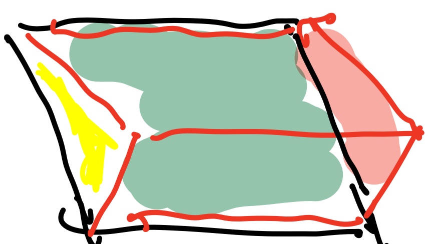
$$(\vec{V}_1 + \vec{V}_2) \times \vec{V}_3 = \underbrace{(\text{scalar}) \times (\text{vector})}_{\text{undefined}}$$

Never use $\times$ to mean ordinary multiplication

$$4 \times (3+2) = 4\times 3 + 4\times 2$$

$$4(3+2) = 4(3) + 4(2)$$





Area(Yellow triangle) = Area(pink triangle)

Large black parallelogram = yellow region and green region

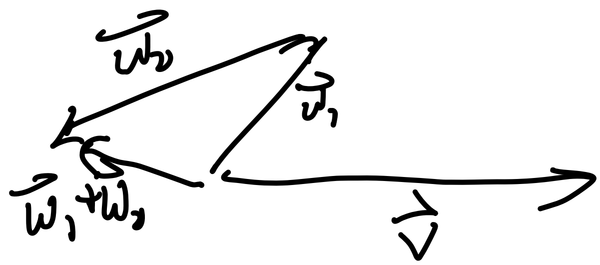
2 smaller red parallelograms = pink region and green region

$\Longrightarrow$ Area(Larger ▱)

$\quad\quad$ ‖

Area(Smaller ▱ + smaller ▱)

$$\vec{v} \times (\vec{w}_1 + \vec{w}_2) = \vec{v} \times \vec{w}_1 + \vec{v} \times \vec{w}_2$$

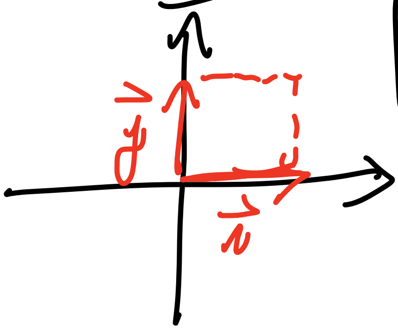


Compare dot product to cross product

Dot product	2D Cross product
$\vec{v} \cdot \vec{w} = \text{scalar}$	$\vec{v} \times \vec{w} = \text{scalar}$
$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$	$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$
$\vec{v} \cdot \vec{v} = \lvert\vec{v}\rvert^2$	$\vec{v} \times \vec{v} = 0$
$(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w})$	$(c\vec{v}) \times \vec{w} = c(\vec{v} \times \vec{w})$
$(\vec{v}_1 + \vec{v}_2) \cdot \vec{w} = \vec{v}_1 \cdot \vec{w} + \vec{v}_2 \cdot \vec{w}$	$(\vec{v}_1 + \vec{v}_2) \times \vec{w} = \vec{v}_1 \times \vec{w} + \vec{v}_2 \times \vec{w}$

Cross product $\Longleftrightarrow$ oriented area of parallelogram

Calculation of cross-product



$$\boxed{\vec{\imath} \times \vec{j} = 1}$$

$$\vec{j} \times \vec{\imath} = -1$$

$$\vec{\imath} \times \vec{\imath} = \vec{j} \times \vec{j} = 0$$

$$(3\vec{\imath} - \vec{j}) \times (\vec{\imath} + \vec{j})$$

$$\langle 3, -1 \rangle \times \langle 1, 1 \rangle =$$

$$(3\vec{\imath} - \vec{j}) \times (\vec{\imath} + \vec{j})$$

$$= (3\vec{\imath} - \vec{j}) \times \vec{\imath} + (3\vec{\imath} - \vec{j}) \times \vec{j}$$

$$= 3\cancel{\vec{\imath} \times \vec{\imath}} - \underbrace{\vec{j} \times \vec{\imath}}_{= -(-1)} + \underbrace{3\vec{\imath} \times \vec{j}}_{= 3} - \cancel{\vec{j} \times \vec{j}}$$

$$= 4 = \det \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} = 3(+1) - (-1)(1) = 4$$

$$(3\vec{i}-\vec{j})\times\vec{i} = 3\vec{i}\times\vec{i} - \cancel{\vec{i}\times\vec{j}}$$

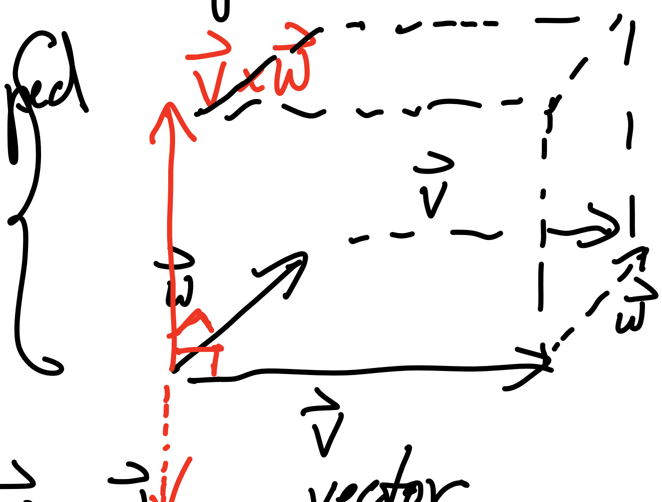
Keep the order of factors unchanged

$$(3x-y)x = 3x^2 - xy \quad \text{OK}$$

$$(3\vec{i}-\vec{j})\times\vec{i} \neq 3\vec{i}\times\vec{i} - \vec{i}\times\vec{j}$$

Cross product in 3-space

parallelopiped



base and top are same parallelogram with sides $\vec{V}, \vec{W}$

Sides are vertical

1) $\vec{V}\times\vec{W}$ = vector

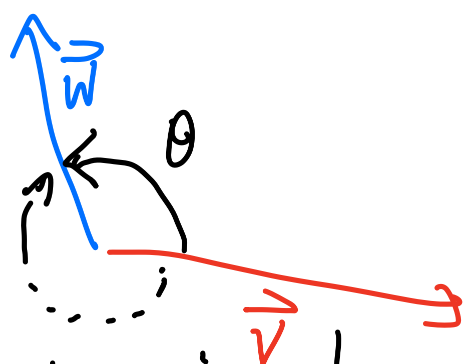
2) $\vec{V}\times\vec{W}$ orthogonal to both $\vec{V},\vec{W}$

$\implies$ orthogonal to plane parallel to both $\vec{V}$ and $\vec{W}$

3) $(\vec{v}, \vec{w}, \vec{v} \times \vec{w})$ has positive orientation

4) Magnitude of $\vec{v} \times \vec{w}$
= volume of parallelopiped with edges $\vec{v}, \vec{w}, \vec{v} \times \vec{w}$

Righthand rule

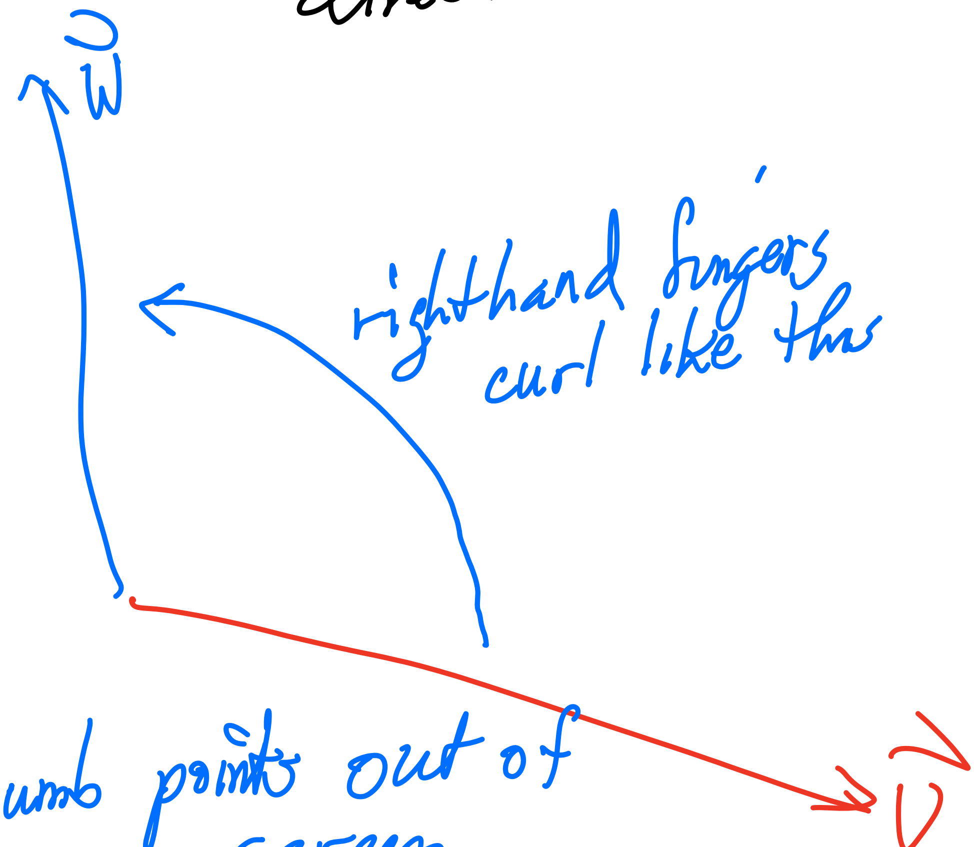


If $\vec{v}, \vec{w}$ lie in plane of the screen (or paper)
then $\vec{v} \times \vec{w}$ is perpendicular to surface of screen
so it either points into screen out of it.

$\vec{v} \times \vec{w}$ has what direction

Curl fingers so they point in direction from $\vec{v}$ towards $\vec{w}$

Direction of thumb is direction of $\vec{v} \times \vec{w}$



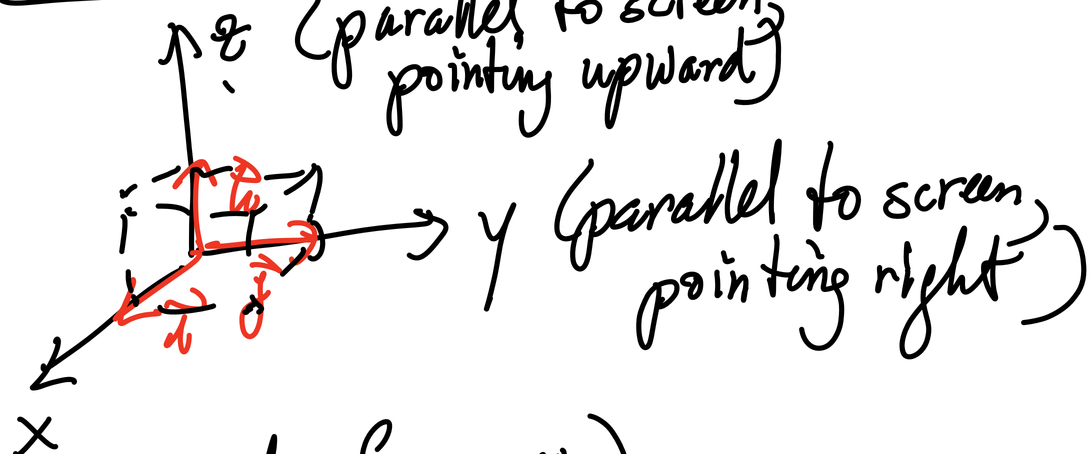
righthand fingers curl like this

Thumb points out of screen

$\Longrightarrow$ $\vec{v} \times \vec{w}$ points out of screen

$\vec{w} \times \vec{v}$ points into screen.

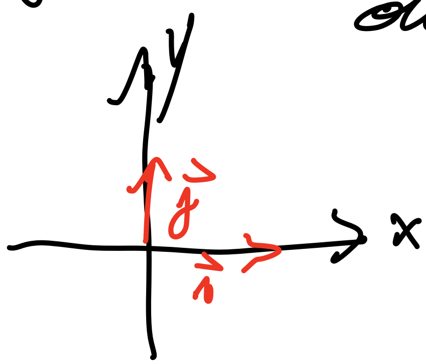
Slides (bottom of slide 16) has link to a video showing 3 different ways to find direction of $\vec{v} \times \vec{w}$



Parallelopiped with edges $\vec{i}, \vec{j}, \vec{k}$ is $1 \times 1 \times 1$ cube so volume is 1 and $(\vec{\imath}, \vec{\jmath}, \vec{k})$ has positive orientation

$$\vec{\imath} \times \vec{\jmath} = \vec{k}$$

Another perspective:
(xy)-plane is surface of screen
Look at it from above
(z-axis points straight out of screen)



z-axis points outward from screen.

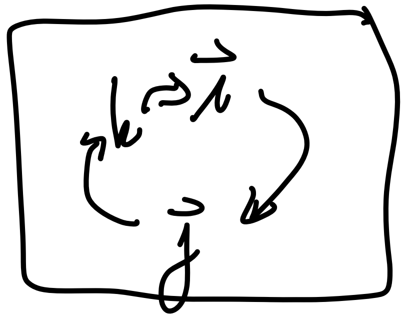
$$\vec{\imath} \times \vec{\jmath} = \vec{k}$$
$$\vec{\jmath} \times \vec{k} = \vec{\imath}$$
$$\vec{k} \times \vec{\imath} = \vec{\jmath}$$

Basic formulas for cross product

1) $\vec{i} \times \vec{j} = \vec{k}$

2) $\vec{j} \times \vec{k} = \vec{i}$

3) $\vec{k} \times \vec{i} = \vec{j}$



Properties of cross product in 3-space

1) $\vec{V} \times \vec{W}$ = vector

2) $(\vec{V}, \vec{W}, \vec{V} \times \vec{W})$ obeys right hand rule
= has positive orientation

3) $\vec{V} \times \vec{W} = -(\vec{W} \times \vec{V})$

4) $\vec{V} \times \vec{V} = 0$

5) $(\vec{V}_1 + \vec{V}_2) \times \vec{W} = \vec{V}_1 \times \vec{W} + \vec{V}_2 \times \vec{W}$

$\vec{V} \times (\vec{W}_1 + \vec{W}_2) = \vec{V} \times \vec{W}_1 + \vec{V} \times \vec{W}_2$

6) $(c\vec{V}) \times \vec{W} = \vec{V} \times (c\vec{W}) = c(\vec{V} \times \vec{W})$

$$(\vec{i} + 2\vec{j} + 3\vec{k}) \times (5\vec{i} + 7\vec{j} + 11\vec{k})$$

$= \cdot \; - -$ long calculation

$\vec{i} \times \vec{j} = \vec{k}$

$\vec{j} \times \vec{k} = \vec{i}$

$\boxed{\vec{k} \times \vec{i} = \vec{j}}$

$$(\vec{i} + \vec{j} + \vec{k}) \times \vec{k}$$

$$= \vec{i} \times \vec{k} + \vec{j} \times \vec{k} + \vec{k} \times \vec{k}$$

$$= -\vec{j} + \vec{i} + \vec{0}$$

$$= \vec{i} - \vec{j}$$

(please write answer in the right order

$(\;\;)\vec{i} + (\;\;)\vec{j} + (\;\;)\vec{k}$

$$(\vec{k} - \vec{i}) \times (\vec{j} + \vec{k})$$

$$= (\vec{k} - \vec{i}) \times \vec{j} + (\vec{k} - \vec{i}) \times \vec{k}$$

$$= \vec{k} \times \vec{j} - \vec{i} \times \vec{j} - \vec{i} \times \vec{k}$$

$$= -\vec{i} - \vec{k} + \vec{j}$$

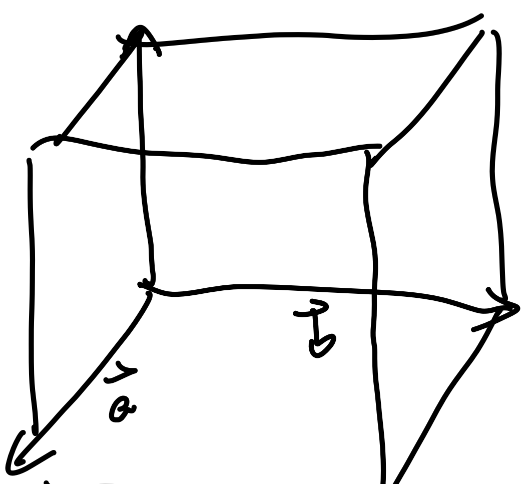
$$= -\vec{i} + \vec{j} - \vec{k}$$

$$\vec{k} - \vec{i} = \langle -1, 0, 3 \rangle$$

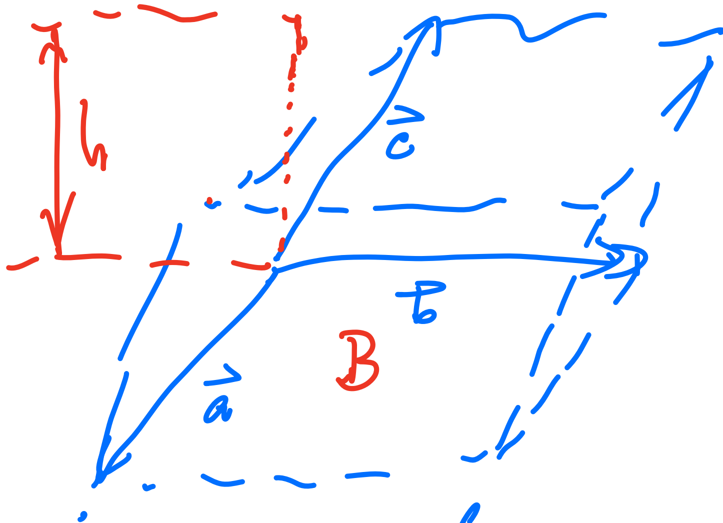
$$\vec{j} + \vec{k} = \langle 0, 1, 1 \rangle$$

$$\langle -1, 0, 3 \rangle \times \langle 0, 1, 1 \rangle$$

$$= \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} \vec{i} & \vec{j} \\ -1 & 0 \\ 0 & 1 \end{matrix}$$



$|\vec{a} \times \vec{b}|$ = volume of this parallelopiped



$\vec{a}, \vec{b}, \vec{c}$ edges of a parallelopiped

Volume = ?

V = (area of B)(h)