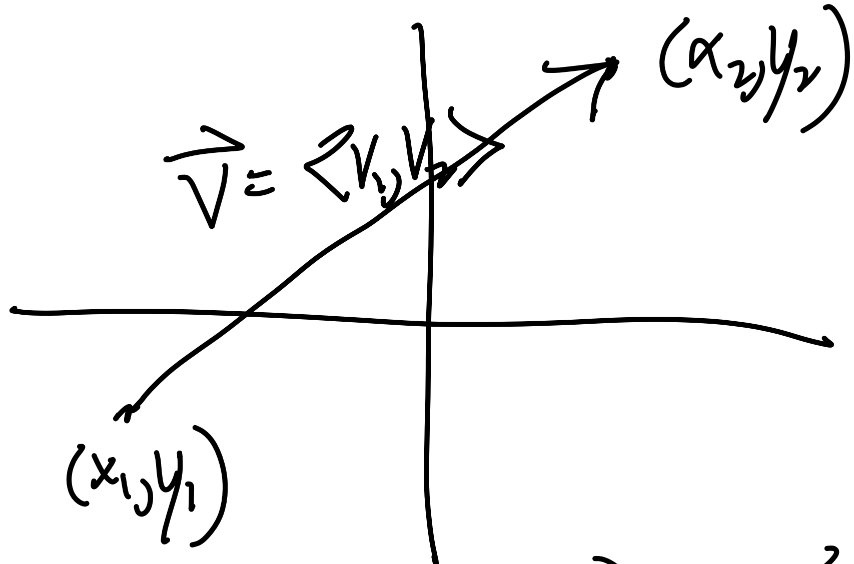
Vectors and Points



$$(x_2, y_2) - (x_1, y_1) = \langle v_1, v_2 \rangle$$

$$|\langle v_1, v_2 \rangle|^2 = v_1^2 + v_2^2$$

$$\shortparallel$$

$$d((x_1, y_1), (x_2, y_2))$$

$$\shortparallel$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2$$

Similar formulas for 3-space element

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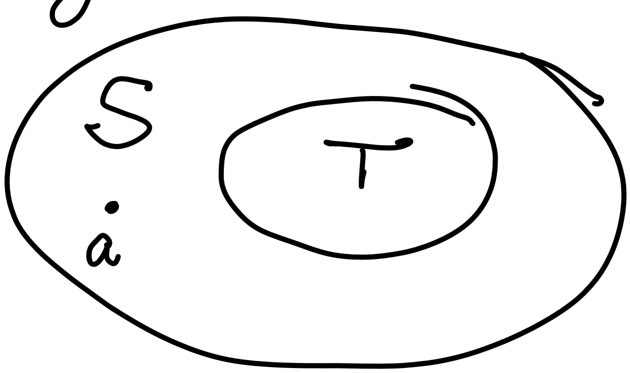
Notation: If S is a set (i.e., collection of things)

$\boxed{a \in S}$ means "a is an element of S"

If T is a set, $T \subset S$ means "everything in T is in S" } "T is a subset of S"

Venn diagrams



Basic property of length

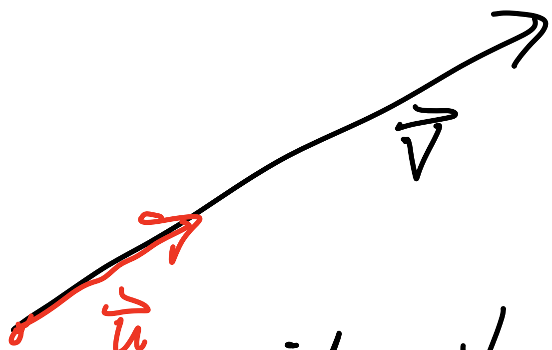
$$|s\vec{v}| = |s|\,|\vec{v}|$$

In particular if $\vec{v} \neq 0$ and $\vec{u} = \dfrac{\vec{v}}{|\vec{v}|}$

$$\frac{\vec{v}}{|\vec{v}|} \text{ means } \left(\frac{1}{|\vec{v}|}\right)\vec{v}$$

and
$$\left|\frac{\vec{v}}{|\vec{v}|}\right| = \underbrace{\boxed{\frac{1}{|\vec{v}|}}}_{s}\,\vec{v} = \frac{1}{|\vec{v}|}|\vec{v}|$$
$$= 1$$

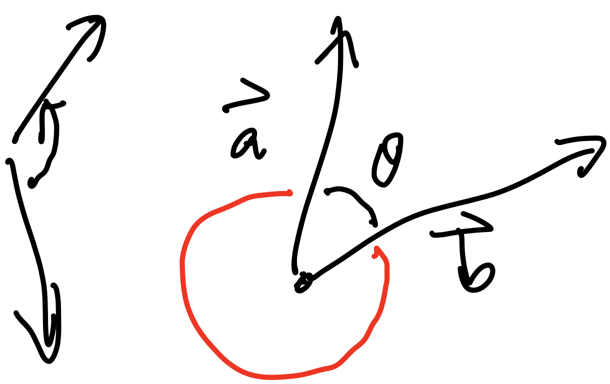
So $\vec{u} = \dfrac{\vec{v}}{|\vec{v}|}$ is a unit vector (means length is 1) that points in same direction as $\vec{v}$



Sometimes a unit vector is called a direction

2 vectors $\vec{V}_1, \vec{V}_2$ are parallel if either they point in same direction or they point in opposite directions

$$\Longleftrightarrow \frac{\vec{V}_1}{|\vec{V}_1|} = \pm \frac{\vec{V}_2}{|\vec{V}_2|} \quad \left(\text{Assuming } \vec{V}_1, \vec{V}_2 \neq 0\right)$$

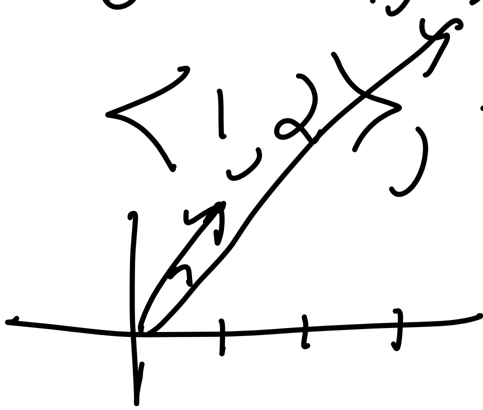


Work with angle between 2 vectors

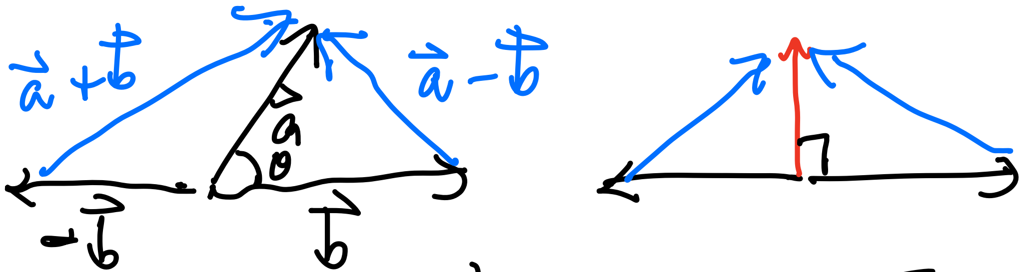
$\vec{a} = \langle a_1, a_2 \rangle$; $\vec{b} = \langle b_1, b_2 \rangle$

Hard to calculate θ from the vectors

$\langle 1, 2 \rangle$, $\langle 3, 4 \rangle$



Work with angles indirectly



$$|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}| \iff \theta = \frac{\pi}{2}$$

$$|\vec{a}-\vec{b}| < |\vec{a}+\vec{b}| \implies 0 \le \theta < \frac{\pi}{2}$$

$$|\vec{a}-\vec{b}| > |\vec{a}+\vec{b}| \implies \frac{\pi}{2} < \theta \le \pi$$

As θ goes from 0 to π,

$|\vec{a}+\vec{b}|^2 - |\vec{a}-\vec{b}|^2$

starts positive, decreases to 0 at $\frac{\pi}{2}$ and then decreases to negative values.

Strictly decreasing function

It "measures" the angle θ

$$|\vec{a}+\vec{b}|^2 - |\vec{a}-\vec{b}|^2$$

$$= (a_1+b_1)^2 + (a_2+b_2)^2 - \left[(a_1-b_1)^2 + (a_2-b_2)^2\right]$$

$$= \cancel{a_1^2} + 2a_1b_1 + \cancel{b_1^2} + \cancel{a_2^2} + 2a_2b_2 + \cancel{b_2^2} - \left[\cancel{a_1^2} - 2a_1b_1 + \cancel{b_1^2} + \cancel{a_2^2} - 2a_2b_2 + \cancel{b_2^2}\right]$$

$$= 4(a_1b_1 + a_2b_2)$$

$a_1b_1 + a_2b_2$ is a way to measure the angle θ

So this is called the dot product

$$\langle a_1, a_2\rangle \cdot \langle b_1, b_2\rangle = a_1b_1 + a_2b_2$$

(vector) $\cdot$ (vector) = scalar

$\langle 1, 2 \rangle \cdot \langle 3, -4 \rangle$

$= 1(3) + (2)(-4) = 3 - 8$

$= -5$

Already know:

If $\vec{a}, \vec{b} \neq \vec{0}$, then

$\vec{a} \cdot \vec{b} = 0 \Longrightarrow$ angle θ between them is $\frac{\pi}{2}$

$\vec{a} \cdot \vec{b} > 0 \Longrightarrow 0 \leq \theta < \frac{\pi}{2}$

$\vec{a} \cdot \vec{b} < 0 \Longrightarrow \frac{\pi}{2} < \theta \leq \pi$

Same holds in 3-space

$\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle$

$= a_1 b_1 + a_2 b_2 + a_3 b_3$

Same properties hold

The dot "$\cdot$" now, has a new specific meaning

$\vec{v} \cdot \vec{w}$ = dot product of $\vec{v}, \vec{w}$ using formulas given above

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Please do not use it in the traditional sense of scalar multiplication

$(3+4)\cdot(5-1)$ ← Don't use this anymore

$(3+4)(5-1)$ ← OK

Properties of dot product

1) $\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2$

$= |\vec{a}|^2 \geq 0$

and $= 0 \iff \vec{a} = \vec{0}$

2) $(-\vec{a}) \cdot \vec{a} = -|\vec{a}|^2$

3) Remember $\vec{a} \cdot \vec{b} = 0$,

this does NOT imply $\vec{a} = \vec{0}$

or $\vec{b} = \vec{0}$

It just means that $\underbrace{\vec{a} \perp \vec{b}}_{\theta = \frac{\pi}{2}}$

4) Algebraic properties

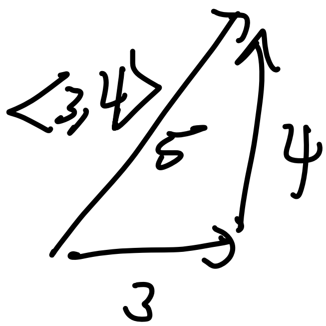
a) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ — Commutative

$a_1 b_1 + a_2 b_2 = b_1 a_1 + b_2 a_2$

b) $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$ — ordinary multiplication

1) $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

$$\langle 3,4\rangle \cdot \langle 3,4\rangle = 3^2 + 4^2$$
$$= 9 + 16 = 25$$
$$= 5^2$$



$\vec{a} \cdot \vec{a} = 0$ means $a_1^2 + a_2^2 = 0$

This means $a_1 = a_2 = 0$

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4 b) $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$ bilinear

$\vec{a} \cdot (c\vec{b}) = c(\vec{a} \cdot \vec{b})$

but note

$$(2\vec{v}) \cdot (2\vec{w}) = 4\, \vec{v} \cdot \vec{w}$$

Factor scalars out of dot product carefully.

4c) Distributivity

$$(\vec{a}_1 + \vec{a}_2) \cdot \vec{b} = \vec{a}_1 \cdot \vec{b} + \vec{a}_2 \cdot \vec{b}$$

$$\vec{a} \cdot (\vec{b}_1 + \vec{b}_2) = \vec{a} \cdot \vec{b}_1 + \vec{a} \cdot \vec{b}_2$$

Examples

i) Find a nonzero vector orthognal to $\langle 13, 17 \rangle$

$\vec{v} = \langle a, b \rangle$ is orthognal to $\langle 13, 17 \rangle$ if

$$\vec{v} \cdot \langle 13, 17 \rangle = 0$$

$$a13 + b17 = 0$$

Choose a, b so that this holds

$a = 17$, $b = -13$ works

$a=1$, $b=\frac{-13}{17}$ works

2) $\langle 1,1\rangle$, $\langle -1,3\rangle$

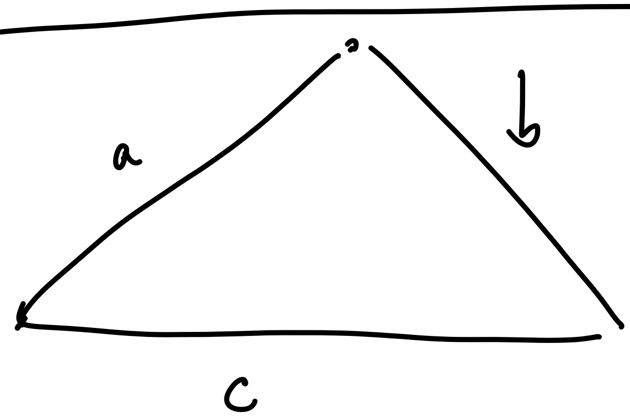
Is angle between these vectors acute or obtuse

$\left(\theta<\frac{\pi}{2}\right)$ $\left(\theta>\frac{\pi}{2}\right)$

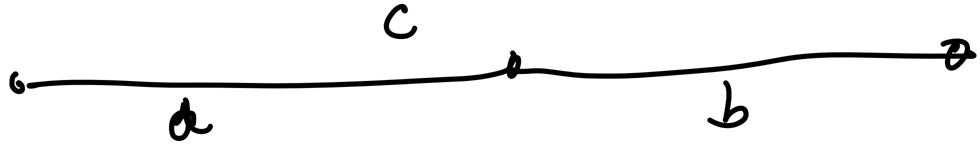
$$\langle 1,1\rangle\cdot\langle -1,3\rangle$$
$$= -1+3 = 2 > 0$$

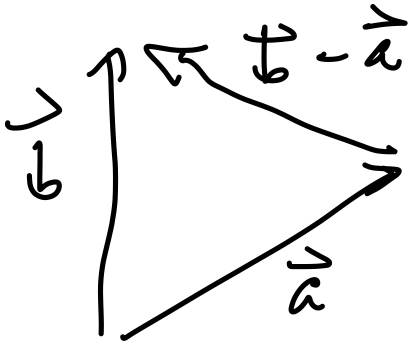
$\Rightarrow$ angle is $<\frac{\pi}{2}$

$\Rightarrow$ angle is acute



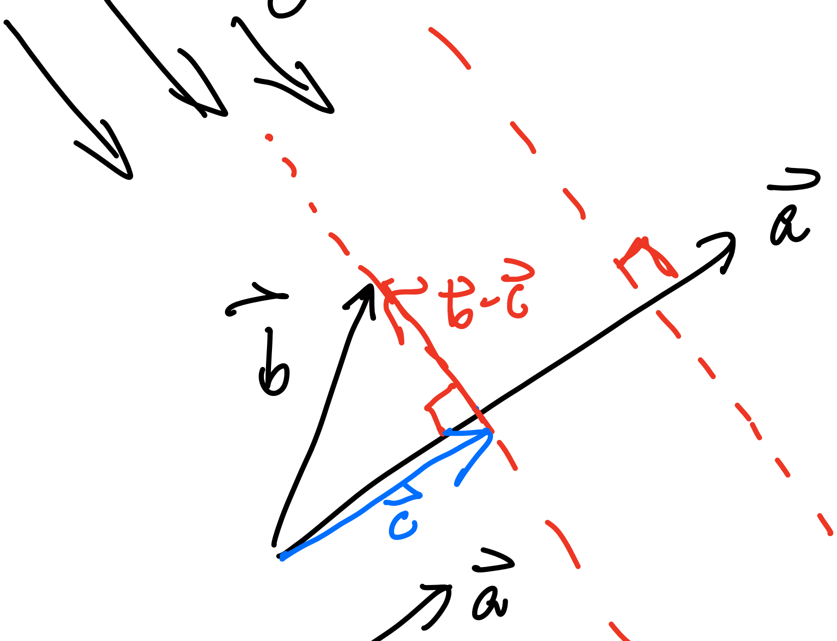
$$\boxed{c < a+b}$$

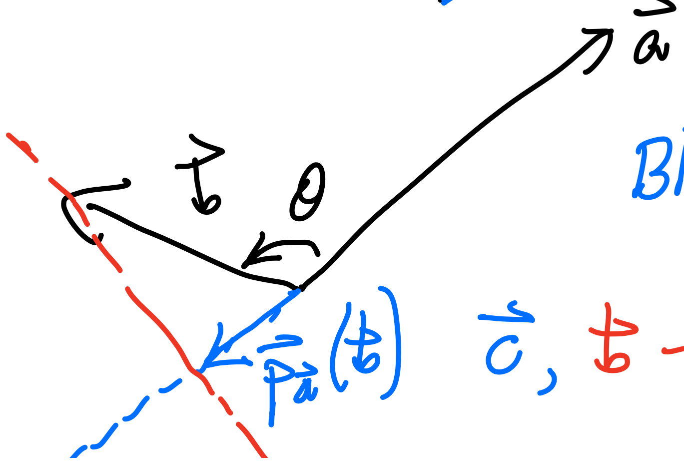




$$|\vec{b}-\vec{a}| < |\vec{a}| + |\vec{b}|$$

Orthogonal projection





Blue vector parallel to $\vec{a}$

$\vec{c}$ parallel to $\vec{a}$

$\vec{c}$, $\vec{b}-\vec{c}$ are orthogonal

$$\vec{b} = \vec{c} + (\vec{b} - \vec{c})$$

parallel　　orthogonal

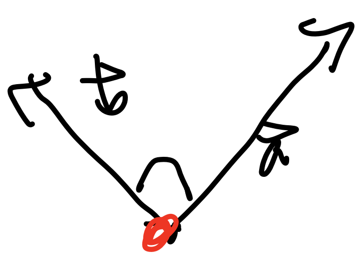
$\vec{c}$ is shadow of $\vec{b}$ on line through $\vec{a}$

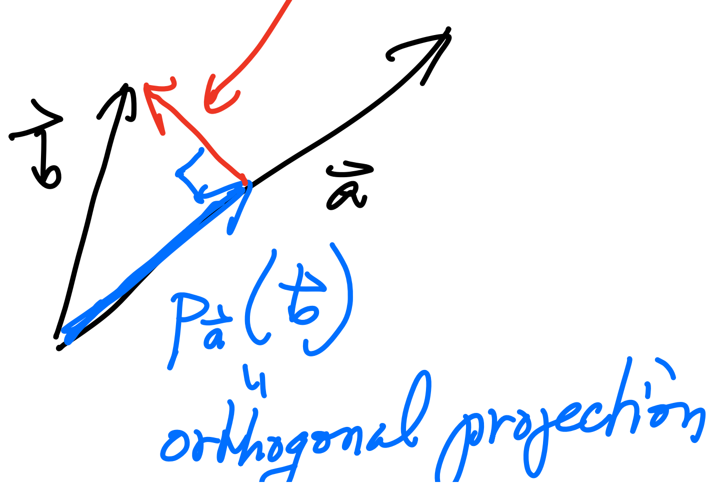
$\vec{c}$ is called the orthogonal projection of $\vec{b}$ onto line through $\vec{a}$

Notation

$$\vec{c} = P_{\vec{a}}(\vec{b})$$

$\vec{b} - P_{\vec{a}}(\vec{b})$





So $\left(\vec{b} - p_{\vec{a}}(\vec{b})\right) \cdot p_{\vec{a}}(\vec{b}) = 0$

Properties θ = angle between $\vec{a}, \vec{b}$

1) $\theta < \frac{\pi}{2} \Rightarrow \vec{a}, p_{\vec{a}}(\vec{b})$ same direction

2) $\theta > \frac{\pi}{2} \Rightarrow \vec{a}, p_{\vec{a}}(\vec{b})$ opposite directions

3) $\theta = \frac{\pi}{2} \Rightarrow p_{\vec{a}}(\vec{b}) = \vec{0}$

4) $\theta = 0 \Rightarrow p_{\vec{a}}(\vec{b}) = \vec{b}$

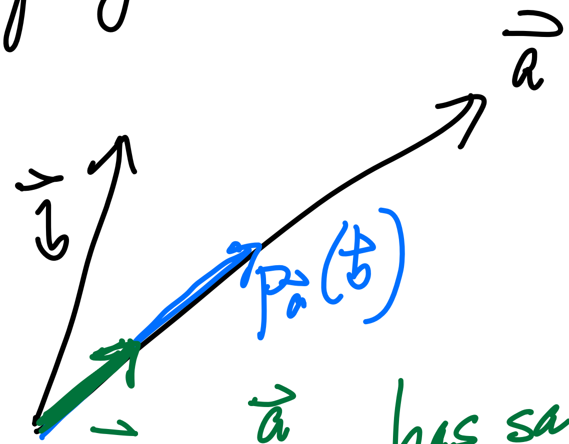
5) $|p_{\vec{a}}(\vec{b})| \leqslant |\vec{b}|$

$\vec{0} = \langle 0, 0, 0 \rangle$

$\neq$

0

Scalar projection



$\vec{u} = \dfrac{\vec{a}}{|\vec{a}|}$ has same direction as $\vec{a}$

$P_{\vec{a}}(\vec{b})$ has same or opposite direction as $\vec{u}$

So there is a scalar factor $c_{\vec{a}}(\vec{b})$ such

$$P_{\vec{a}}(\vec{b}) = \left(c_{\vec{a}}(\vec{b})\right)\vec{u}$$

$c_{\vec{a}}(\vec{b})$ called scalar projection

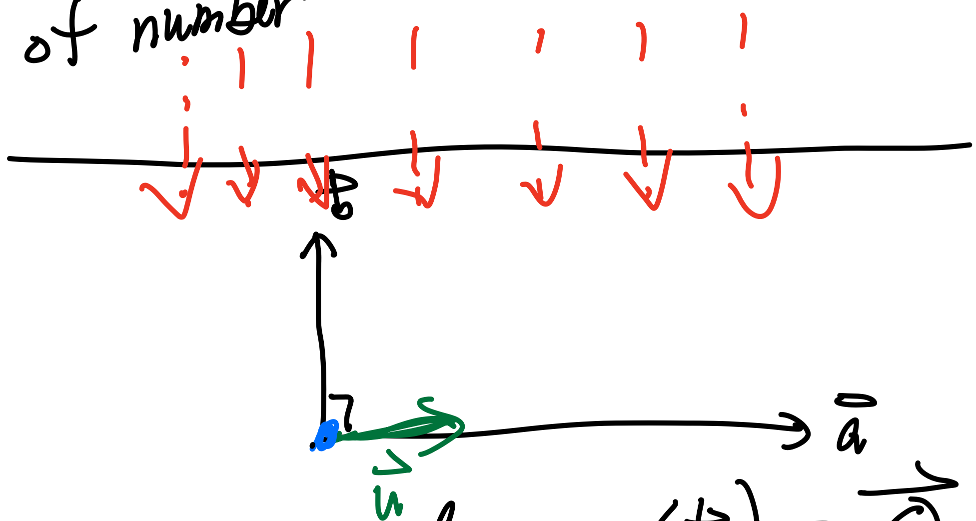
If angle θ between $\vec{a}, \vec{b}$ is $< \frac{\pi}{2}$, then $c_{\vec{a}}(\vec{b}) > 0$

If $\theta > \frac{\pi}{2}$, $c_{\vec{a}}(\vec{b}) < 0$

$$\left| C_{\vec{a}}(\vec{b}) \right| = \left| \vec{P}_{\vec{a}}(\vec{b}) \right|$$

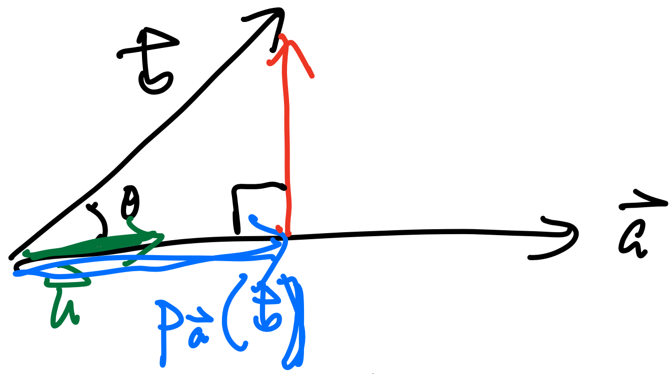
absolute value of number

length of vector



If $\vec{b} \perp \vec{a}$, then $P_{\vec{a}}(\vec{b}) = \vec{0}$

$$C_{\vec{a}}(\vec{b}) = 0$$



$$p_{\vec{a}}(\vec{b}) = c_{\vec{a}}(\vec{b})\,\vec{u}$$

Since $\theta < \frac{\pi}{2}$, $c_{\vec{a}}(\vec{b}) > 0$

$$\Rightarrow \quad c_{\vec{a}}(\vec{b}) = \left|p_{\vec{a}}(\vec{b})\right|$$

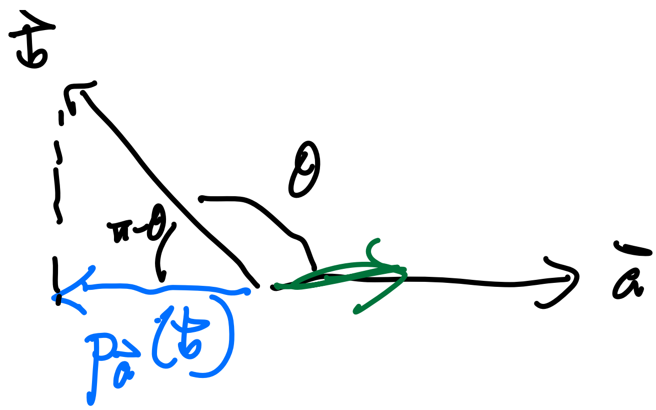
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\|$

length of adjacent side

$|\vec{b}|$ = length of hypotenuse

$$\cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$$

$$\cos\theta = \frac{c_{\vec{a}}(\vec{b})}{|\vec{b}|} \Rightarrow c_{\vec{a}}(\vec{b}) = |\vec{b}|\cos\theta$$

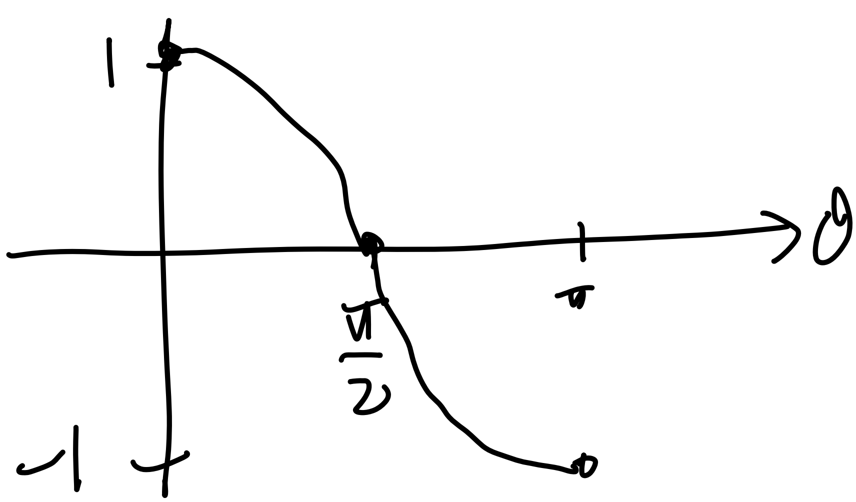


$$c_a(\vec{b}) < 0$$

$$c_a(\vec{b}) = |\vec{b}| \cos\theta$$

still holds because if $\theta > \frac{\pi}{2}$,

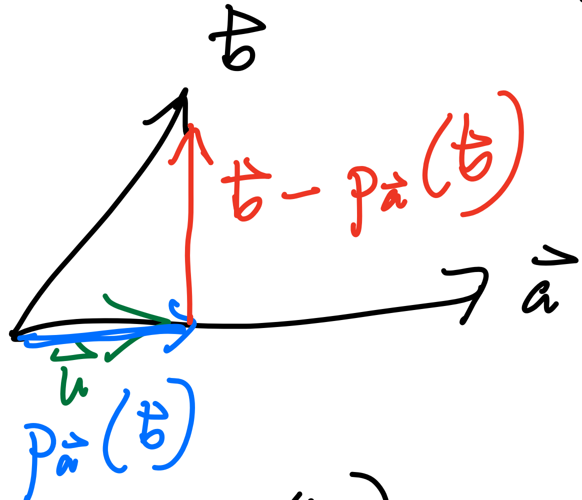
$$\cos\theta < 0$$



So

$$c_{\vec{a}}(\vec{b}) = |\vec{b}| \cos\theta$$ always holds for any angle

Scalar projection using dot product



Since $\vec{u} \perp \vec{b} - p_{\vec{a}}(\vec{b})$,

$$
\begin{aligned}
0 &= \vec{u} \cdot (\vec{b} - p_{\vec{a}}(\vec{b})) \\
&= \vec{u} \cdot \vec{b} - \vec{u} \cdot p_{\vec{a}}(\vec{b}) \\
&= \vec{u} \cdot \vec{b} - \vec{u} \cdot \left(c_{\vec{a}}(\vec{b})\,\vec{u}\right) \\
&= \vec{u} \cdot \vec{b} - \left(c_{\vec{a}}(\vec{b})\right) \vec{u} \cdot \vec{u} \\
&= \vec{u} \cdot \vec{b} - \left(c_{\vec{a}}(\vec{b})\right) \underbrace{|\vec{u}|^2}_{=1} \\
&= \vec{u} \cdot \vec{b} - c_{\vec{a}}(\vec{b})
\end{aligned}
$$

$$\Longrightarrow \quad c_{\vec a}(\vec b) = \vec b \cdot \vec u$$

$$\underset{|\vec b|\cos\theta}{\parallel}$$

$$P_{\vec a}(\vec b) = c_{\vec a}(\vec b)\,\vec u$$
$$= (\vec b \cdot \vec u)\,\vec u$$

$\vec a \neq 0$

$\vec u = \dfrac{\vec a}{|\vec a|}$

$P_{\vec a}(\vec b) = (\vec b \cdot \vec u)\,\vec u$

$c_{\vec a}(\vec b) = \boxed{\vec b \cdot \vec u = |\vec b|\cos\theta}$

Formula for dot product

$$\vec a = |\vec a|\,\vec u \Longrightarrow \vec a \cdot \vec b = (|\vec a|\vec u)\cdot \vec b$$
$$= |\vec a|\,(\vec u \cdot \vec b)$$

$$\vec{a} \cdot \vec{b} = \left(|\vec{a}|\vec{u}\right) \cdot \vec{b}$$

$$= |\vec{a}|\left(\vec{u} \cdot \vec{b}\right)$$

$$= |\vec{a}||\vec{b}|\cos\theta$$

$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta}$$

$$\cos(-\theta) = \cos\theta$$

Use only if absolutely necessary

$$c_{\vec{a}}(\vec{b}) = \vec{b}\cdot\vec{u}$$
$$= \vec{b}\cdot\left(\frac{\vec{a}}{|\vec{a}|}\right) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

$$P_{\vec{a}}(\vec{b}) = c_{\vec{a}}(\vec{b})\vec{u}$$
$$= (\vec{b}\cdot\vec{u})\vec{u}$$
$$= \frac{(\vec{b}\cdot\vec{a})\vec{a}}{|\vec{a}|^2}$$

To compute projections, it's best to

1) Find $\vec{u} = \frac{\vec{a}}{|\vec{a}|}$

2) Use formulas

Properties of orthogonal projection

1) a) Linear

$$P_{\vec{a}}(\vec{b}_1+\vec{b}_2) = P_{\vec{a}}(\vec{b}_1) + P_{\vec{a}}(\vec{b}_2)$$

$$\parallel$$

$$((\vec{b}_1+\vec{b}_2)\cdot\vec{u})\vec{u}$$

$$\parallel$$

$$(\vec{b}_1\cdot\vec{u} + \vec{b}_2\cdot\vec{u})\vec{u}$$

$$\parallel$$

$$(\vec{b}_1\cdot\vec{u})\vec{u} + (b_2\cdot\vec{u})\vec{u}$$

$$\parallel$$

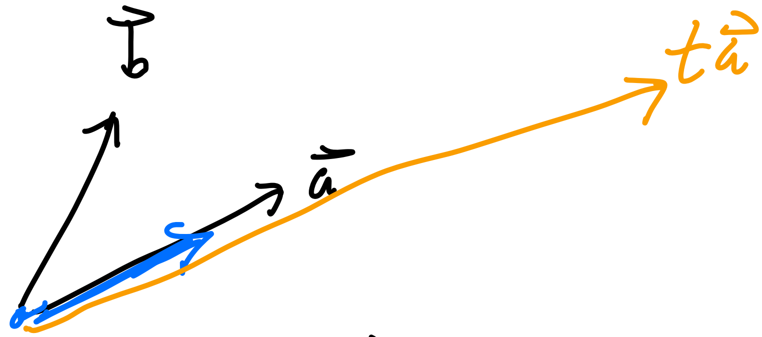
$$P_{\vec{a}}(\vec{b}_1) + P_{\vec{a}}(\vec{b}_2)$$

b)

$$P_{\vec{a}}(t\vec{b}) = t\,P_{\vec{a}}(\vec{b})$$

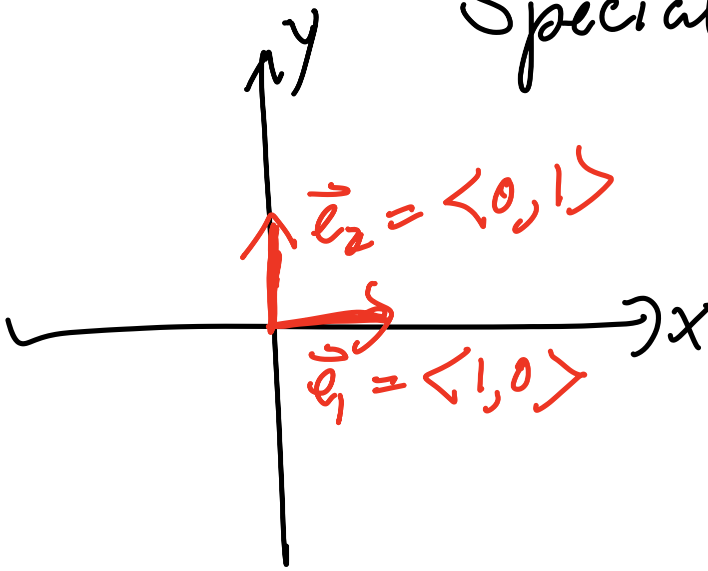
2) $t > 0$

$$P_{t\vec{a}}(\vec{b}) = P_{\vec{a}}(\vec{b})$$



$$P_{-t\vec{a}}(\vec{b}) = -P_{\vec{a}}(\vec{b})$$

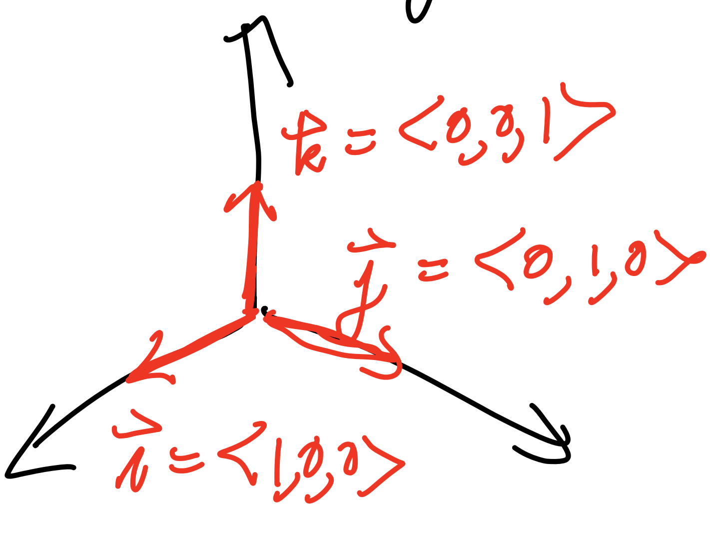
Special vectors



$$\vec{v} = \langle v_1, v_2 \rangle = v_1 \vec{e}_1 + v_2 \vec{e}_2$$

Also write $\vec{i} = \vec{e}_1 = \langle 1,0 \rangle$

$\vec{j} = \vec{e}_2 = \langle 0,1 \rangle$



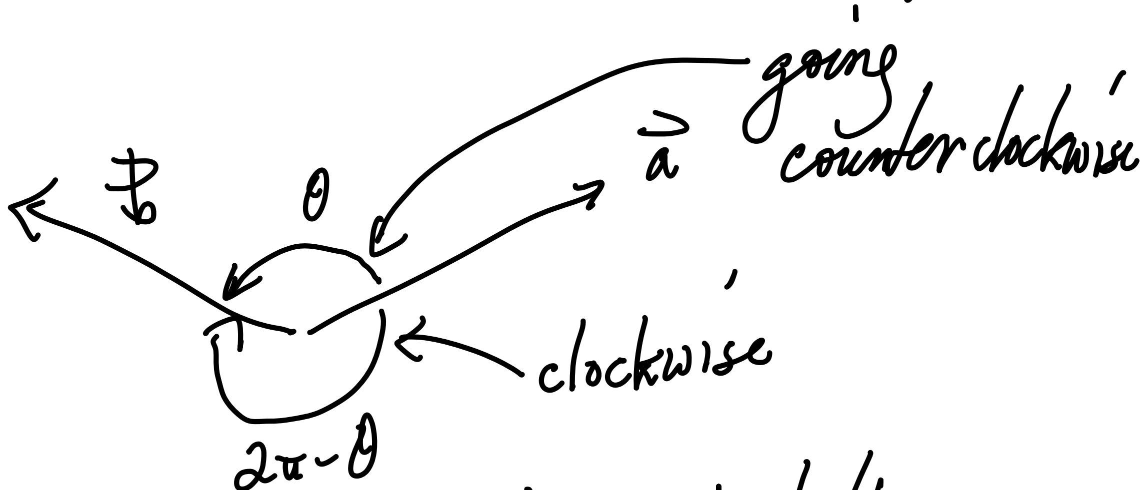
$$|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$$

$$\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$$

$$\vec{v} = \langle v_1, v_2 \rangle = v_1 \vec{i} + v_2 \vec{j}$$
$$= \vec{i}\, v_1 + \vec{j}\, v_2$$

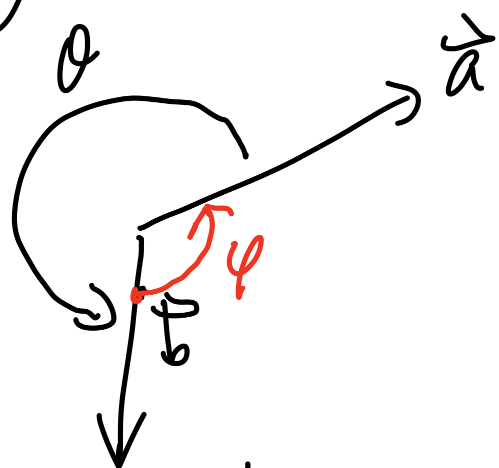
Orientation (order matters)

Suppose $\vec{a}, \vec{b}$ are nonzero non parallel vectors in 2-space



$(\vec{a}, \vec{b})$ has positive orientation if counterclockwise angle θ is $0 \le \theta \le \pi$

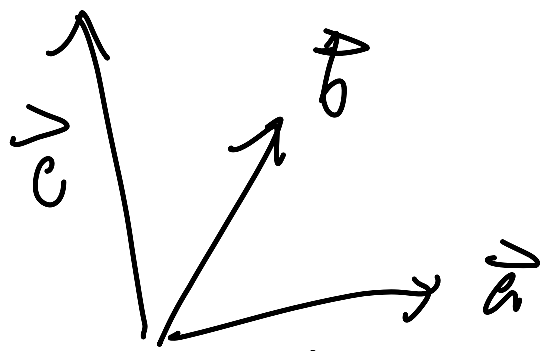
and negative orientation otherwise



Here $(\vec{a}, \vec{b})$ has negative orientation

but $(\vec{b}, \vec{a})$ has positive orientation

Next time orientation of an ordered triple of vectors in 3-space



$$(\vec{a}, \vec{b}, \vec{c})$$

$$\begin{array}{ccc} \hat{i}, & \hat{j}, & \hat{k} \\ \shortparallel & \shortparallel & \shortparallel \\ \vec{i} & \vec{j} & \vec{k} \end{array}$$