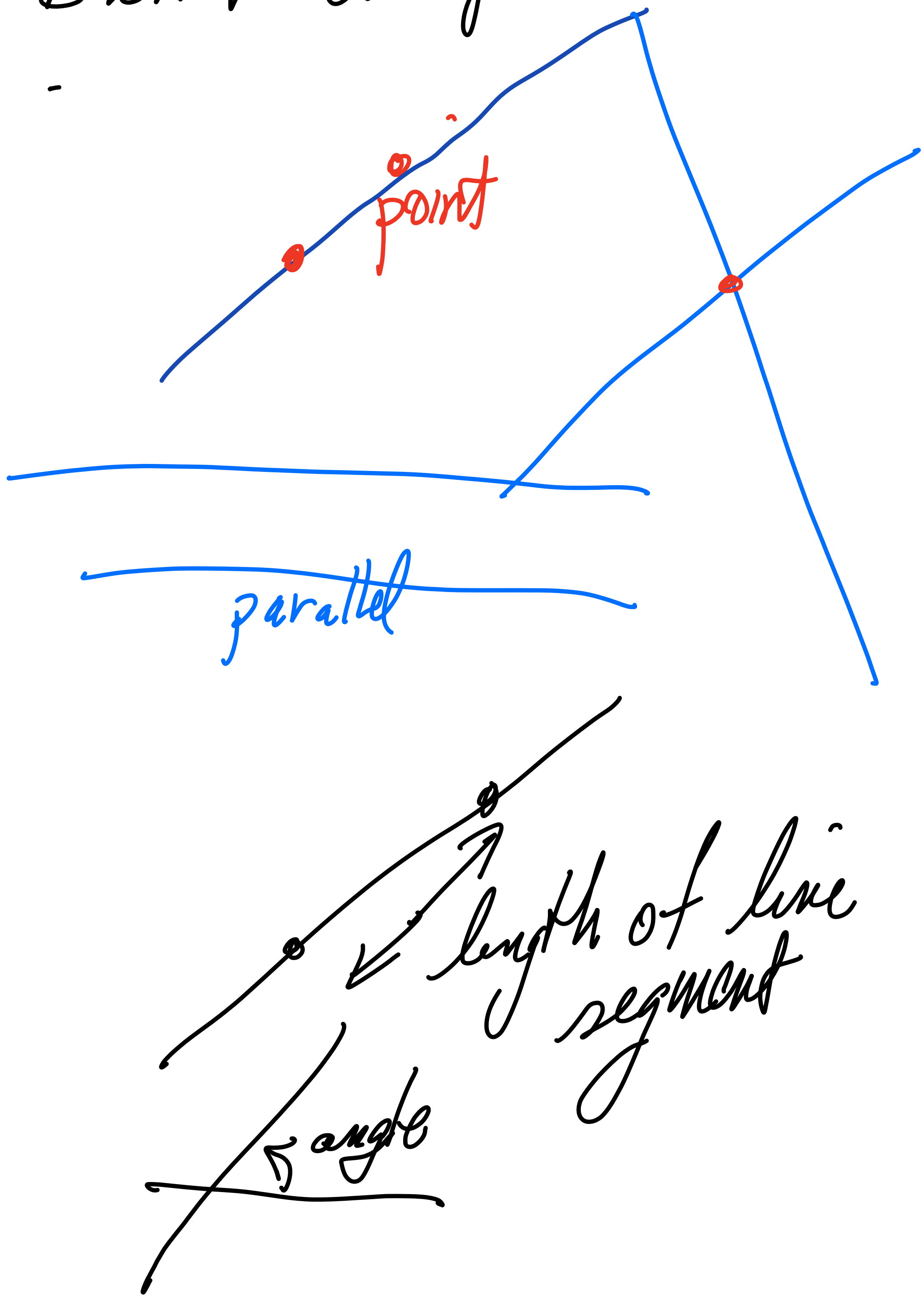
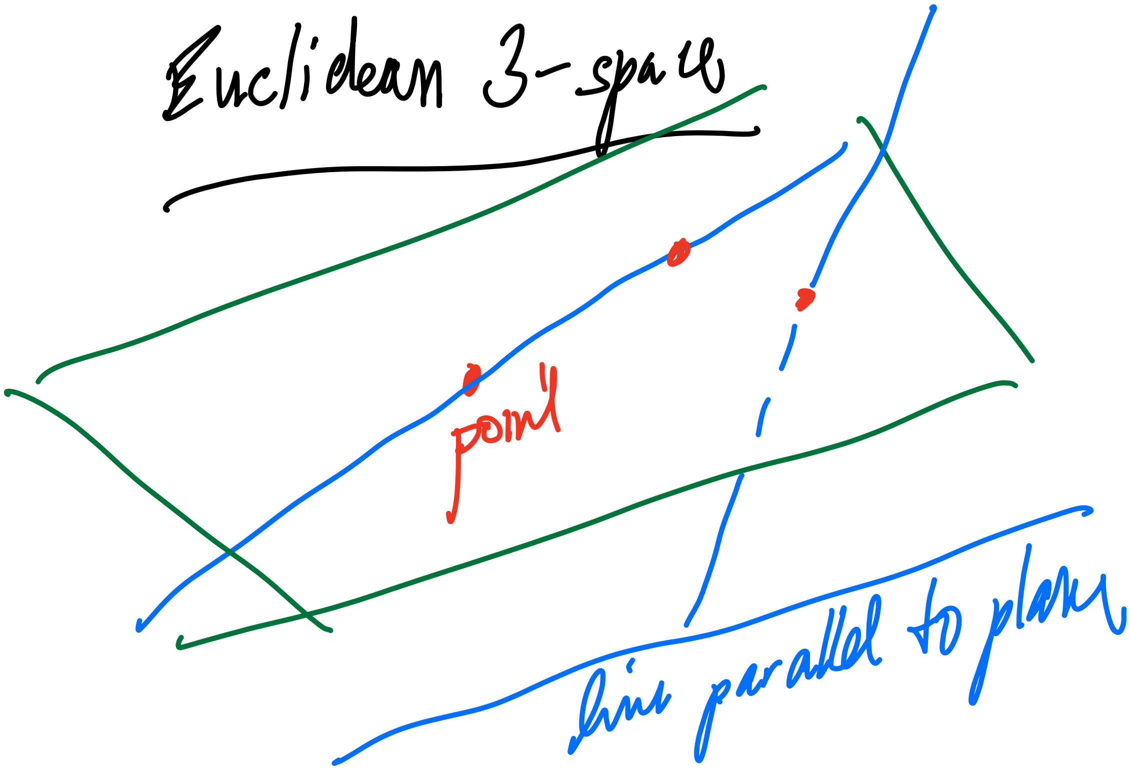
# Eucliden 2-space

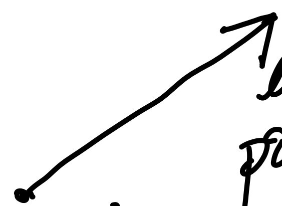


# Euclidean 3-space



# Vectors in 2-space and 3-space

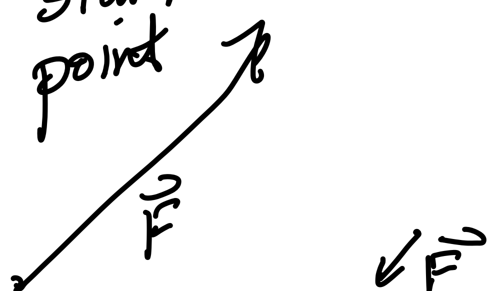
Vector = arrow : line segment with a direction and length



end point

start point

magnitude

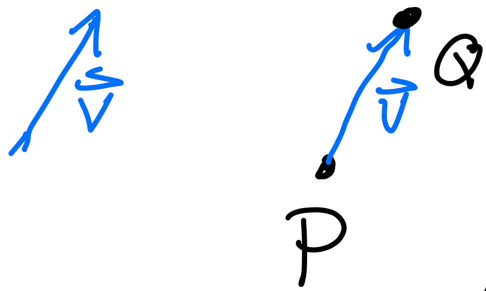



larger magnitude than this

A vector has no particular location in space

# What can we do with vectors?

Start with a point, put start of vector at P and get a new point, which is endpoint of $\vec{v}$

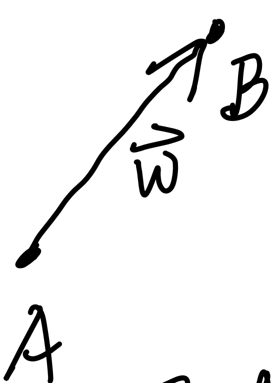



Q is P translated by $\vec{v}$

Write this as addition

point + vector = new point

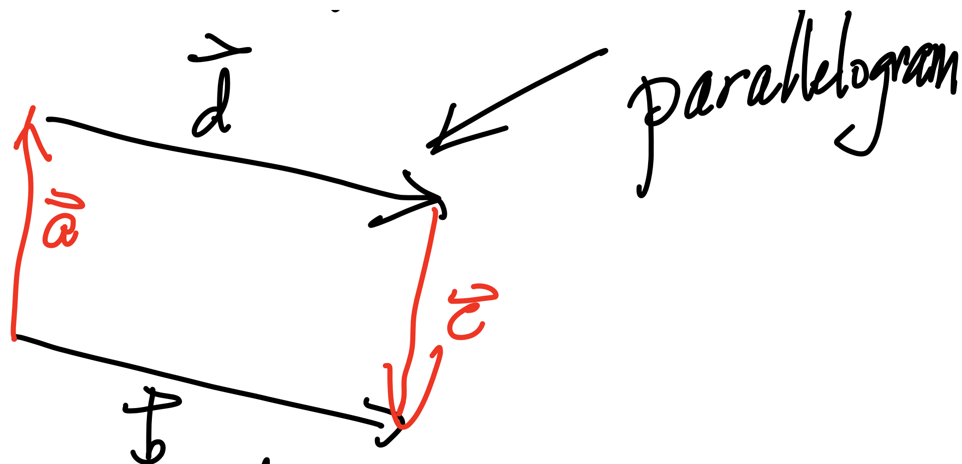
$$P + \vec{v} = Q$$




$$A + \vec{w} = B$$

$$\underbrace{B - A = \vec{v}}$$

$B - A$ = vector from point A to point B




$\vec{a}, \vec{c}$ parallel

$\vec{b}, \vec{d}$ parallel, same direction, same length

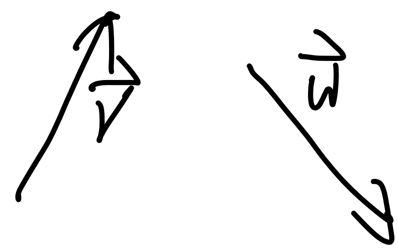
$\Longrightarrow \vec{b} = \vec{d}$

$\vec{a}, \vec{c}$ same length but opposite directions

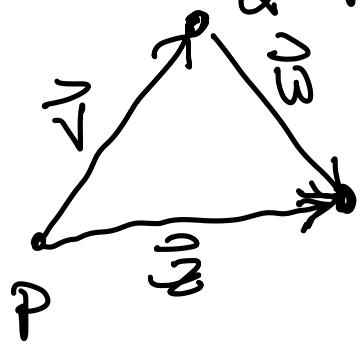
Notation: $-\vec{a}$ = vector with same length but opposite direction as $\vec{a}$

Here, $\vec{c} = -\vec{a}$

Add vectors



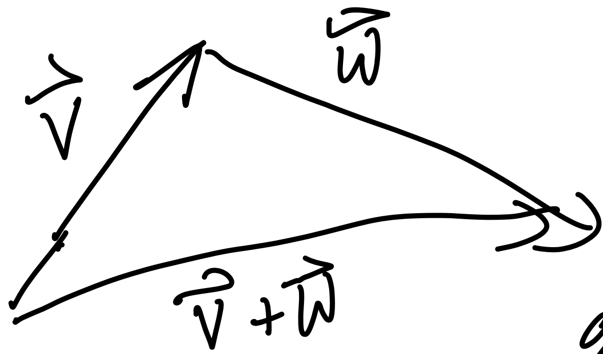

$Q = P + \vec{v}$




$R = Q + \vec{w}$

$= (P + \vec{v}) + \vec{w}$

$\vec{z} = \vec{v} + \vec{w}$

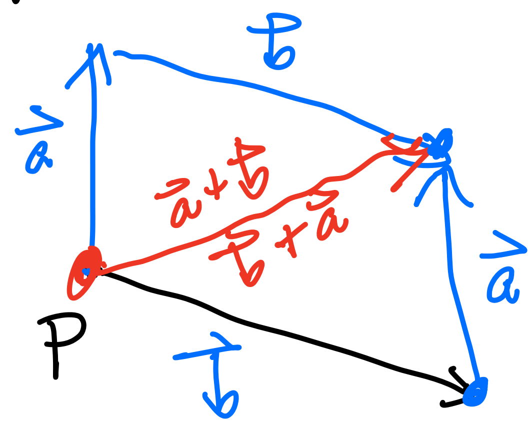



$R = Q + \vec{w}$

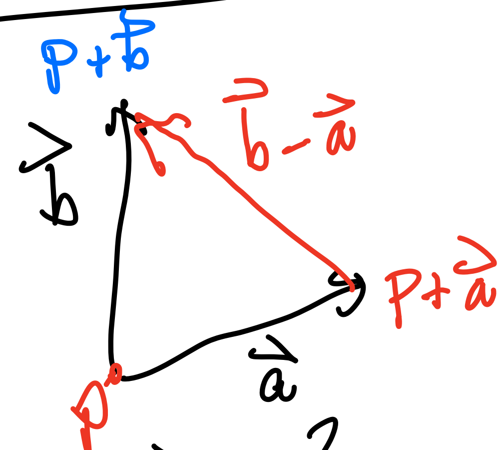
$= (P + \vec{v}) + \vec{w}$ ← get to R in 2 steps

$= P + (\vec{v} + \vec{w})$ ← get to R in 1 step

## Vector addition is commutative




## Vector subtraction



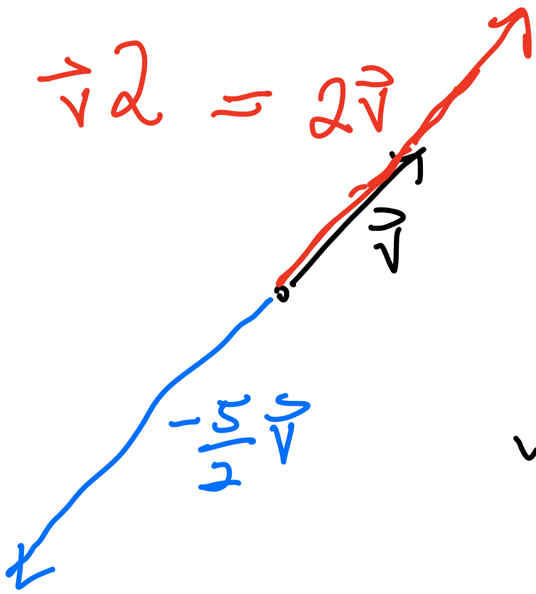

$(b-a)+a=b$

$\vec{b}-\vec{a}=\,?$

Want $\boxed{\vec{a}+(\vec{b}-\vec{a})=\vec{b}}$

$\vec{b}-\vec{a}=$ vector from end of $\vec{a}$ to end of $\vec{b}$

# Rescale a vector




$\vec{v}c = c\vec{v} = \vec{v}$ rescaled by factor c

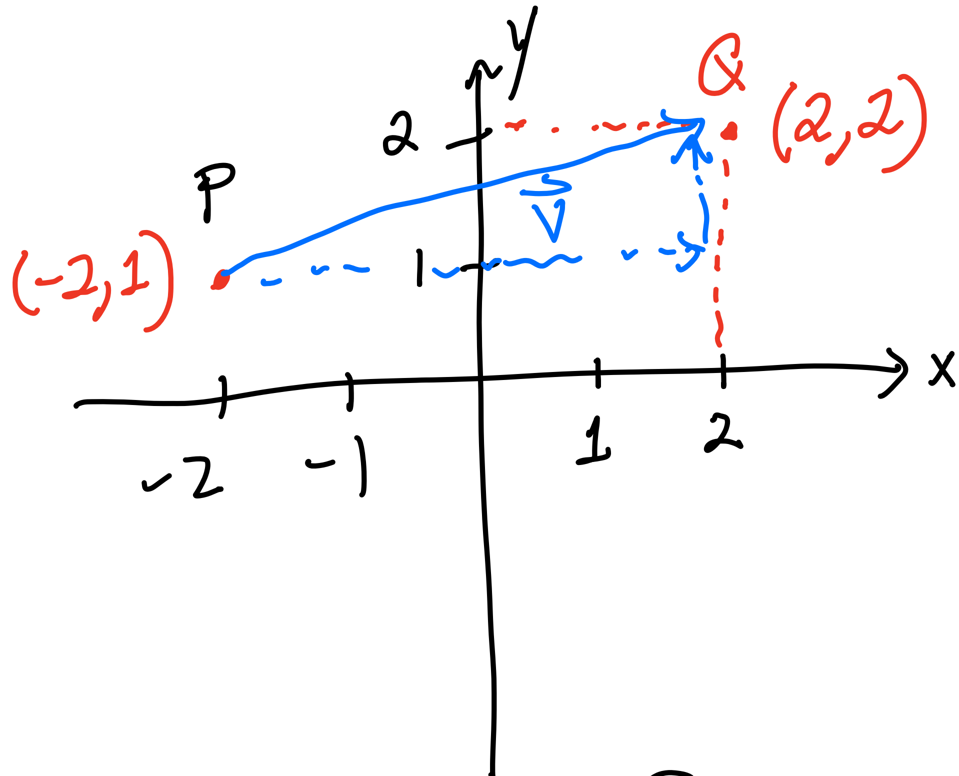
scalar

"scaling"

Descartes

Use coordinates to do Euclidean geometry

Replaces picture by equations and formulas




$\underbrace{P + \vec{v}}$ $= Q$

1) Shift right by 4 units
2) Shift up by 1 unit

4 = change in x-coordinate

1 = change in y-coordinate

Write $\vec{v} = \langle 4, 1 \rangle$

= "add 4 to x-coordinate and 1 to y-coordinate"

$$\vec{V} = \langle V_1, V_2 \rangle$$

By definition

$$P + \vec{V} = \langle x_0 + V_1, y_0 + V_2 \rangle$$

$P = (x_0, y_0)$ $\quad$ $\vec{V} = \langle V_1, V_2 \rangle$

$$(s, t) + \langle a, b \rangle = (s + a, t + b)$$

$(s,t) =$ point

In 3-space

$$P = (s, t, u)$$

$$\vec{V} = \langle a, b, c \rangle$$

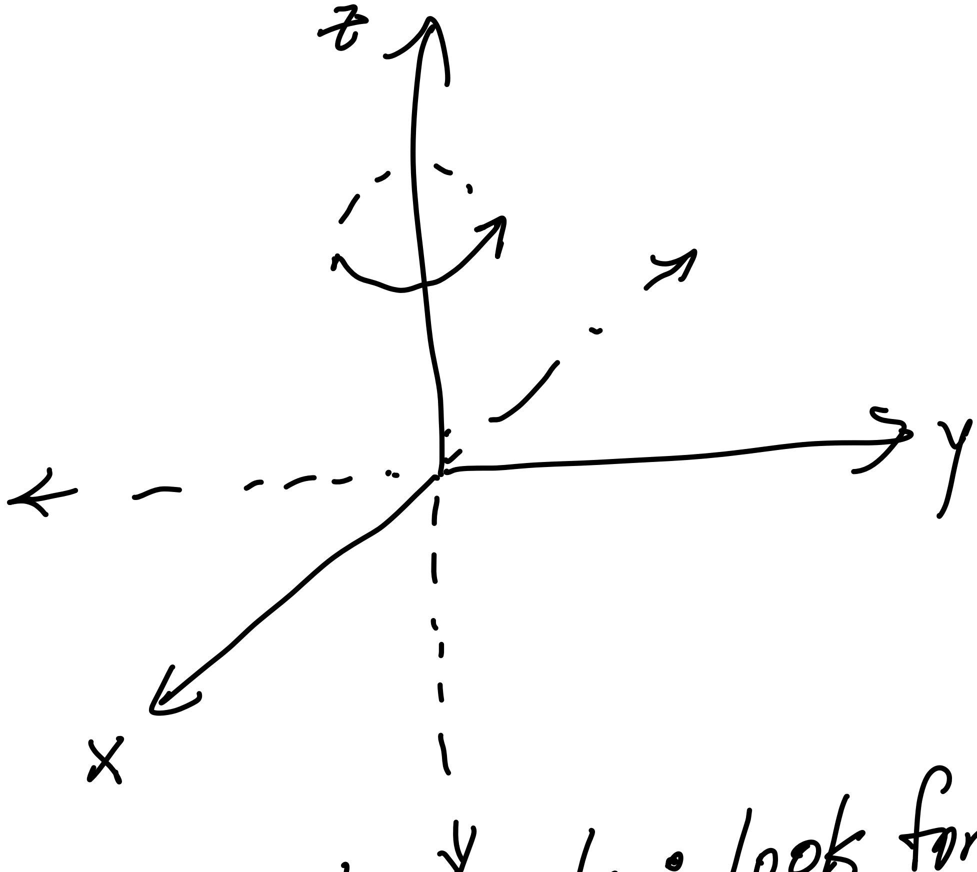
$$P + \vec{V} = (s + a, t + b, u + c)$$

$s, t, u, a, b, c$ are real numbers

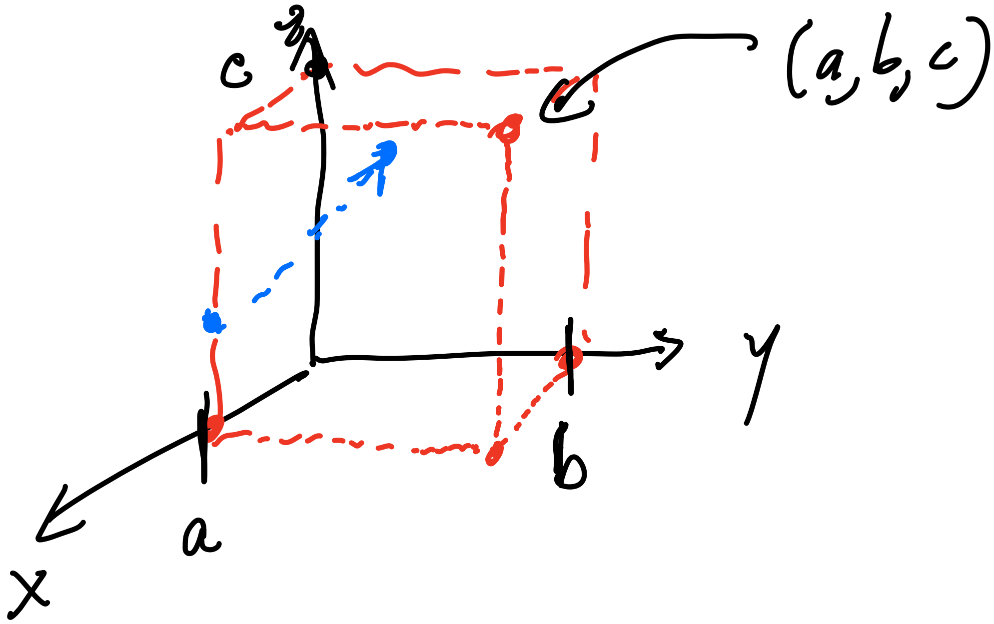
$P$, $P + \vec{V}$ are points

$\vec{V}$ is a vector

Coordinates in 3-space




Righthand rule: look for videos online




Vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$

= endpoint − start point

( , , ) − ( , , )

(1, 2, 3)

(−1, 0, 2)

$(1,2,3) - (-1,0,2)$

$= \underbrace{\langle 2, 2, 1 \rangle}$

add 2 to x-coordinate
add 2 to y-coordinate
add 1 to z-coordinate

Algebra of points, vectors, scalars

(scalar)(scalar) = scalar

(scalar) + (scalar) = scalar

(scalar) - (scalar) = scalar

(scalar)(vector) = vector

vector + vector = vector

vector - vector = vector

point + vector = point

point - point = vector

Not ok

scalar + vector

scalar + point

(scalar)(point)

(vector)(vector)

(vector)(point)

(point)(point)

$$5\langle 1,-1,2\rangle = \langle 5,-5,10\rangle$$

$$5\,(1,-1,2) \neq (5,-5,10)$$

Properties of vector arithmetic

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

$$= \vec{a} + \vec{b} + \vec{c}$$

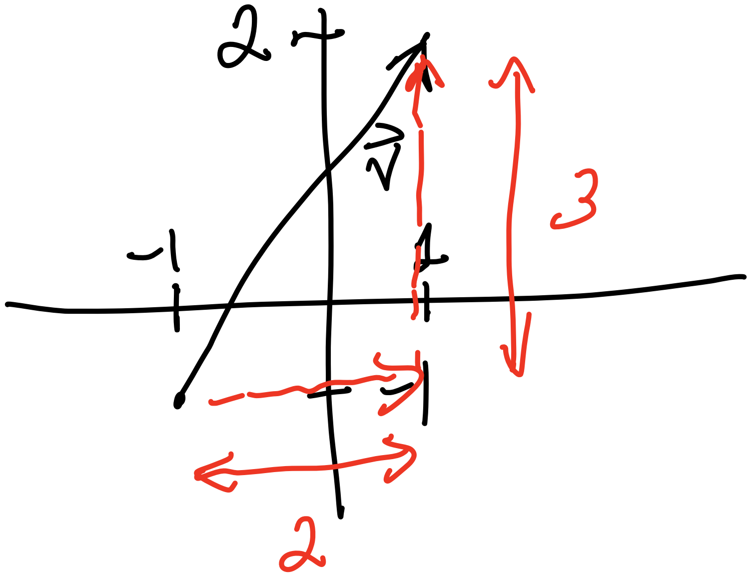
$$(st)\vec{a} = s(t\vec{a})$$

$$= st\vec{a}$$

$$(s+t)\vec{a} = s\vec{a} + t\vec{a}$$

$$s(\vec{a} + \vec{b}) = s\vec{a} + t\vec{b}$$

# Length of vector




$$\vec{v} = \langle 2, 3 \rangle$$
$$= \langle 2, 0 \rangle + \langle 0, 3 \rangle$$

Pythagorean theorem

$$(\text{length of } \vec{v})^2 = 2^2 + 3^2$$
$$= 4 + 9 = 13$$

Notation:

$$|\vec{v}| = \text{length of } \vec{v}$$

$|\text{scalar}|$ = absolute value

$|\text{vector}|$ = length or magnitude

$$(*) \quad \boxed{|\langle a,b\rangle|^2 = a^2+b^2}$$

$$(**) \quad \left( |\langle a,b\rangle| = \sqrt{a^2+b^2} \right)$$

Always try to avoid using $(**)$
(Square roots are nasty)

Try to use $(*)$

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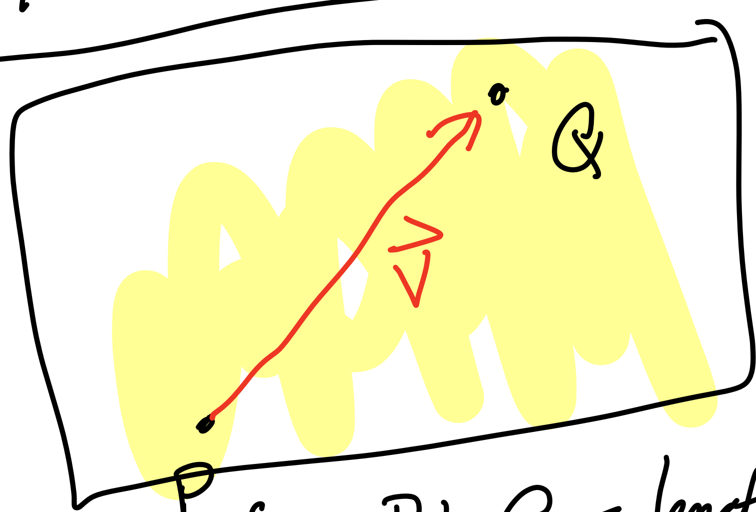



distance from P to Q = length of $\vec{v}$

$$d(P,Q) = |Q-P|$$

$$(d(P,Q))^2 = |Q-P|^2$$

$$P = (p_1, p_2)$$
$$Q = (q_1, q_2)$$

$$(d(P,Q))^2 = ?$$

$$= |Q-P|^2$$

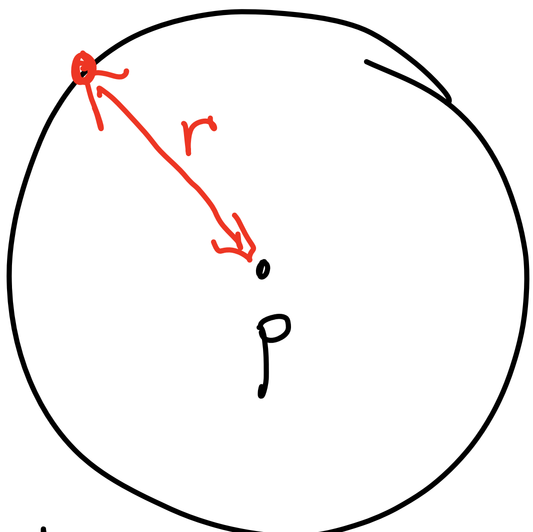
$$= |(q_1, q_2) - (p_1, p_2)|^2$$

$$= |\langle q_1 - p_1, q_2 - p_2 \rangle|^2$$

$$= (q_1 - p_1)^2 + (q_2 - p_2)^2$$

Circle with center $P$
and radius $r$

$$C = \{ \text{points } Q : d(P,Q) = r \}$$




Center $P = (x_0, y_0)$ center

A point $Q = (x,y)$ lies in $C$

if $r^2 = (d(P,Q))^2$

$= |Q - P|^2$

$= |(x,y) - (x_0,y_0)|^2$

$$= |\langle x - x_0, y - y_0 \rangle|^2$$

$$\boxed{r^2 = (x - x_0)^2 + (y - y_0)^2}$$

Equation of circle

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$T$ = circle with center $(e, f)$ and radius $z$

$$= \{ (s, t) : d((s,t), (e,f)) = z \}$$

$$= \{ (s-e)^2 + (t-f)^2 = z^2 \}$$

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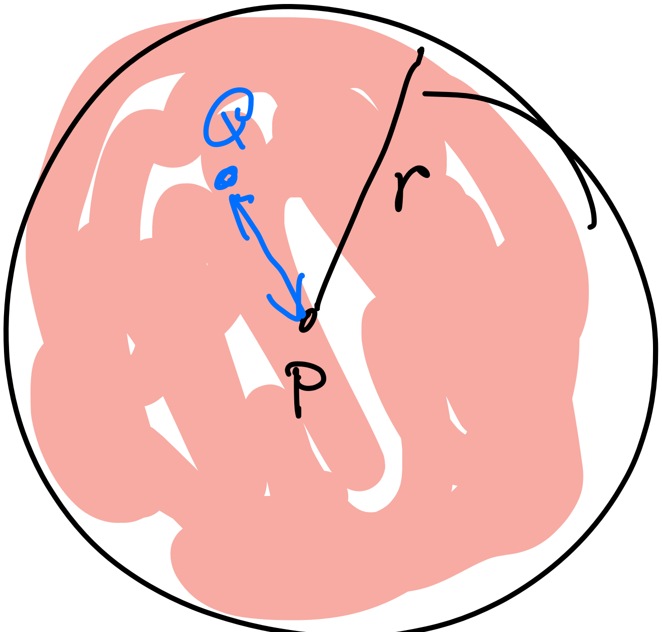
Circle centered at origin

$$x^2 + y^2 = r^2$$

Disk = region inside a circle




D centered at P, radius r

"

{points whose distance to center is less than or equal to r}

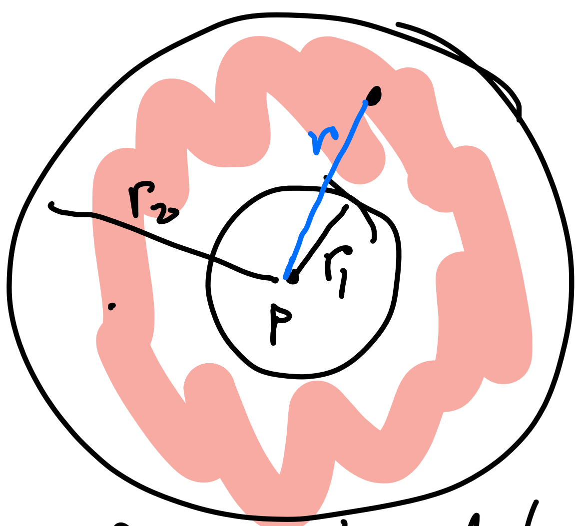
$$D = \{Q : d(P,Q) \leq r\}$$

$$= \{(x,y) : (x-x_0)^2 + (y-y_0)^2 \leq r^2\}$$

$$\left( d(P,Q) \le r \iff (d(P,Q))^2 \le r^2 \right)$$

<u>Annulus</u>

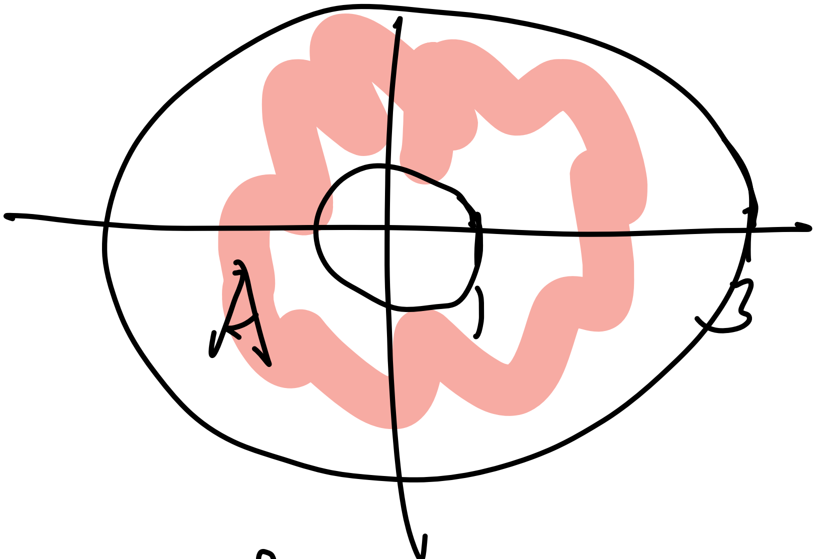
= region inside large disk but outside smaller disk with same center




$0 < r_1 < r_2$

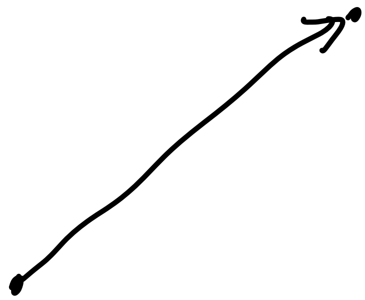
$A = \{$points whose distance to $P$ is $\le r_2$ and $\ge r_1\}$

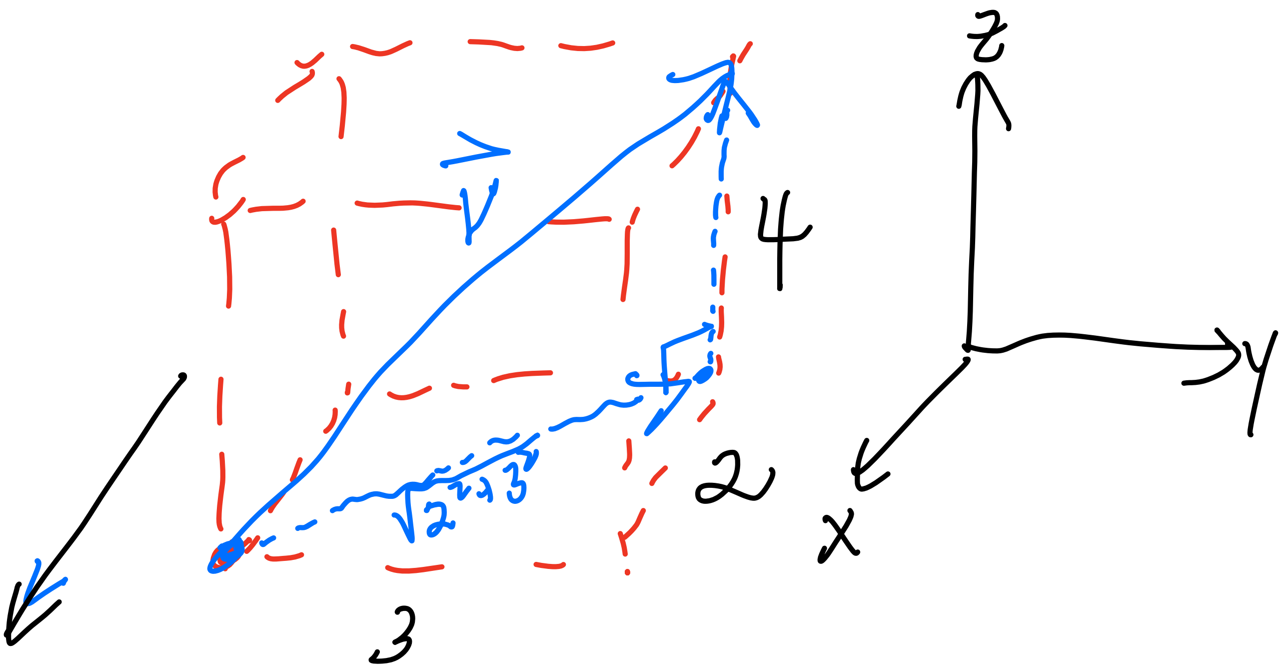
$$= \{ r_1^2 \le (x-x_0)^2 + (y-y_0)^2 \le r_2^2 \}$$




$$A = \{ 1^2 \le x^2 + y^2 \le 3^2 \}$$

Length and distance in 3-space






$$\vec{v} = \cancel{\langle 3, 2, 4 \rangle} \quad \langle -2, 3, 4 \rangle$$

$$|\vec{v}| = \left(\sqrt{2^2+3^2}\right)^2 + 4^2 = 2^2+3^2+4^2$$

$$|\langle v_1, v_2, v_3 \rangle|^2 = v_1^2 + v_2^2 + v_3^2$$

$$d((p_1, p_2, p_3), (q_1, q_2, q_3))$$

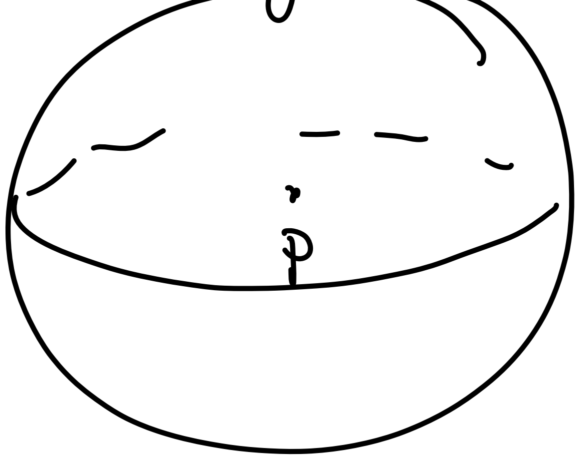
$$= (q_1 - p_1)^2 + (q_2 - p_2)^2 + (q_3 - p_3)^2$$

$$= (p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2$$

Sphere with center $P$, radius $r$

$$S = \{\text{points } Q : d(P, Q) = r\}$$

surface of a ball



Ball = region inside sphere

$= \{ \text{points } Q : d(P,Q) \leq r \}$

Spherical shell center $P$

inner radius $r_1$, $0 < r_1 < r_2$

outer radius $r_2$

$A = \{ Q : r_1 \leq d(P,Q) \leq r_2 \}$

Sphere

$$\{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2\}$$

Ball

$$\{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\leq r^2\}$$

Shell

$$\{r_1^2\leq(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\leq r_2^2\}$$