MATH-GA2450 Complex Analysis

Automorphisms of Disk
Automorphisms of Upper Half Plane
Fractional Linear Transformations

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December 10, 2024

Automorphisms of Unit Disk (Part 1)

- Given $|\alpha| < 1$, let
$$F_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha} z}$$
- If $|z| < 1$, then $|\bar{\alpha} z| < 1$ and therefore $1 - \bar{\alpha} z \neq 0$
- It follows that F_α is analytic on $D = D(0,1)$
- If $|z| = 1$, then $z = e^{i\phi}$, which implies
$$\begin{aligned} F_\alpha(z) &= \frac{e^{i\phi} - \alpha}{1 - \bar{\alpha} e^{i\phi}} \\ &= e^{i\phi} \frac{1 - \alpha e^{-i\phi}}{1 - \bar{\alpha} e^{i\phi}} \\ &= e^{i\phi} \frac{1 - \alpha e^{-i\phi}}{\overline{1 - \alpha e^{i\phi}}}, \end{aligned}$$
  and therefore
$$|F_\alpha(z)| = 1$$
- By the maximum modulous principle, if $|z| < 1$, then $|F_\alpha(z)| < 1$

Automorphisms of Unit Disk (Part 2)

- If
$$w = \frac{z - \alpha}{1 - \bar{\alpha} z},$$
then
$$\begin{aligned} w - \bar{\alpha} w z &= z - \alpha \\ \rightsquigarrow z(1 + \bar{\alpha} w) &= w + \alpha \\ \rightsquigarrow z &= \frac{w + \alpha}{1 + \bar{\alpha} w} \\ &= F_{-\alpha}(w) \end{aligned}$$
- It follows that for each $\alpha \in D$ and $z \in D$,
$$F_\alpha(F_{-\alpha}(z)) = z$$
and therefore, F_α is surjective
- Therefore, $F_\alpha, F_{-\alpha}$ are automorphisms of D

Automorphisms of Unit Disk (Part 3)

- Let $f : D \to D$ be an automorphism of D
- Let $f(\alpha) = 0$
- Then $h = f \circ F_{-\alpha}$ is an automorphism of D such that

$$h(0) = f(F_{-\alpha}(0)) = f(\alpha) = 0$$

- By the Schwarz lemma, if $z \in D$, then

$$|z| < |h(z)|$$

- On the other hand, $h^{-1} = f^{-1} \circ F_{-\alpha}$ is also an automorphism that satisfies $h^{-1}(0) = 0$
- By the Schwarz lemma,

$$|h(z)| < |h^{-1}(h(z))| = |z|$$

Automorphisms of Unit Disk (Part 3)

- Therefore, there exists $\phi \in \mathbb{R}$ such that

$$h(z) = e^{i\phi} z,$$

i.e.,

$$f(F_{-\alpha}(w)) = e^{i\phi} w$$

- Setting $z = F_{-\alpha}(w)$, we get

$$f(z) = e^{i\phi} F_{\alpha}(z) = e^{i\phi} \frac{z - \alpha}{1 - \bar{\alpha} z}$$

Fractional Linear Transformations (Part 1)

- Let $\mathrm{gl}(2, \mathbb{C})$ denote the set of all 2-by-2 complex matrices
- Given

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

  let

$$F_M(z) = \frac{az+b}{cz+d}$$

- The derivative of F is

$$F_M'(z) = \frac{(cz+d)a - (az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}$$

- Therefore, if $\det(M) \neq 0$, then

$$F : \mathbb{C} \setminus \left\{ \frac{-d}{c} \right\} \to F(\mathbb{C} \setminus \left\{ \frac{-d}{c} \right\}$$

  is an analytic isomorphism

Fractional Linear Transformations

- Observe that if $t \in \mathbb{C} \backslash \{0\}$, then $F_{tM} = F_M$
- We can therefore always assume that $\det(M) = 1$
- Let
$$\mathrm{SL}(2, \mathbb{C}) = \{M \in \mathrm{gl}(2, \mathbb{C}) : \det(M) = 1\}$$
- A **fractional linear transformation** is a map F_M where $M \in \mathrm{SL}(2, \mathbb{C})$

Riemann Sphere

- Define **Riemann sphere** to be

$$S = \mathbb{C} \cup \{\infty\}$$

- Any fractional linear transformation F_M can be viewed as a map

$$F_M : S \to S,$$

where if

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$F(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq \frac{-d}{c} \\ \infty & \text{if } z = \frac{-d}{c} \\ \frac{a}{c} & \text{if } z = \infty \end{cases}$$

- Let $\mathcal{F}$ denote the set of all fractional inear transformations

Matrix Multiplication

- If
$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } N = \begin{bmatrix} p & q \\ r & s \end{bmatrix},$$
then
$$MN = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix}$$

The Group of Fractional Linear Transformations

- Given $M, N \in \mathrm{SL}(2, \mathbb{C})$,

$$
\begin{aligned}
(F_M \circ F_N)(z) &= F_M\left(\frac{pz+q}{rz+s}\right) \\
&= \frac{a\left(\frac{pz+q}{rz+s}\right)+b}{c\left(\frac{pz+q}{rz+s}\right)+d} \\
&= \frac{a(pz+q)+b(rz+s)}{c(pz+q)+d(rz+s)} \\
&= \frac{(ap+br)z+(aq+bs)}{(cp+dr)z+(cq+ds)} \\
&= F_{MN}(z)
\end{aligned}
$$

- Therefore, for each $M \in \mathrm{SL}(2, \mathbb{C})$,

$$F_M^{-1} = F_{M^{-1}}$$

- It follows that $\mathcal{F}$ is a group and the map

$$\mathrm{SL}(2, \mathbb{C}) \to \mathcal{F}$$

Decomposition of Fractionsl Linear Transformations

- Basic fractional linear transformations
  - **Scaling:** Given $s \in \mathbb{C}\backslash\{0\}$,
  $$R_s(z) = sz$$
  - **Translation:** Given $\alpha \in \mathbb{C}$
  $$T_\alpha(z) = z + \alpha$$
  - **Inversion:**
  $$J(z) = \frac{1}{z}$$
- **Theorem.** If $F : S \to S$ is a fractional linear transformation, then there exist $s \in \mathbb{C}\backslash\{0\}, \alpha, \beta \in \mathbb{C}$ such that either
  $$F(z) = sz + \beta$$
  or
  $$F = T_\alpha \circ R_s \circ J \circ T_\beta$$

Automorphisms of the Upper Half-Plane

- Assume $a, b, c, d \in \mathbb{R}$ and $ad - bc \neq 0$
- If $z \in \mathbb{R}$, then $F(z) \in \mathbb{R}$ and

$$F(i) = \frac{ai + b}{ci + d} = \frac{(ai + b)(-ci + d)}{c^2 + d^2} = \frac{ac + bd + i(ad - bc)}{c^2 + d^2}$$

- F is undefined when $z = \dfrac{-d}{c}$
- It follows that if $ad - bc > 0$,

$$F : H \to H$$

is an analytic automorphism of H