MATH-GA2450 Complex Analysis

Counting Zeros and Poles
Evaluation of Definite Integrals

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Counting Zeros and Poles Inside Simple Closed Curve

- A closed curve $c : [a, b] \to \mathbb{C}$ is **simple** if for any $z \in \mathbb{C} \backslash c$, the winding number of c around z is 0 or 1
- We say z is inside c if $W(c, z) = 1$
- Let $O \subset \mathbb{C}$ be open and f be a meromorphic function on O with zeros at $a_1, \ldots, a_m$ and poles at $b_1, \ldots, b_n$ inside c
- The **multiplicity of** f **at** z_0 is defined to be

$$\operatorname{mult}_{z_0} f = -\operatorname{ord}_{z_0} f$$

- Then

$$\begin{aligned} \int_c \frac{f'(z)}{f(z)}\,dx &= 2\pi i(p_1 + \cdots + p_m - (q_1 + \cdots + q_n)) \\ &= 2\pi i((\text{number of zeros}) - (\text{number of poles})), \end{aligned}$$

where $p_j = \operatorname{ord}_{a_j} f$ and $q_k = \operatorname{mult}_{b_k} f$ and the numbers of zeros and poles are counted with multiplicity

Rouché's Theorem

- Let $c : [a, b] \to O$ be a simple closed curve in an open $O \subset \mathbb{C}$
- Let f, g be holomorphic functions on O
- If for any $z \in c$,

$$|f(z) - g(z)| < |f(z)|, \tag{1}$$

  then f and g have the same number of zeros inside c

Proof of Rouché's Theorem I

- Observe that (1) implies that neither f nor g have any zeros on c
- Let $F = g/f$
- For each $z \in c$,

$$|F(z) - 1| = \frac{|f(z) - g(z)|}{|f(z)|} < 1$$

  and therefore $F(z) \in D(1, 1)$
- It follows that

$$F \circ c : [a, b] \to \mathbb{C}$$

  is a closed curve in $D(1, 1)$
- Since $0 \notin D(1, 1)$,

$$W(F \circ c, 0) = 0$$

Proof of Rouché's Theorem II

- Therefore,

$$
\begin{aligned}
0 &= W(F \circ c, 0) \\
&= \int_{F \circ c} \frac{dz}{z} \\
&= \int_{t=a}^{t=b} \frac{(F \circ c)'(t)}{F \circ c(t)}\, dt \\
&= \int_{t=a}^{t=b} \frac{F'(c(t))}{F \circ c(t)} c'(t)\, dt \\
&= \int_{c} \frac{F'(z)}{F(z)}\, dz \\
&= \int_{c} \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}\, dz \\
&= (\text{number of zeros of } f) - (\text{number of zeros of } g)
\end{aligned}
$$

Computation of Residues (Part 1)

- If f has a pole at z_0 and g is holomorphic at z_0, then

$$\mathrm{Res}_{z_0}(fg) = g(z_0)\,\mathrm{Res}_{z_0}(f)$$

  - Observe that

$$\begin{aligned} f(z) &= \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots \\ g(z) &= b_0 + b_1(z - z_0) + \cdots \\ f(z)g(z) &= (\frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots)(b_0 + b_1(z - z_0) + \cdots) \\ &= \frac{a_{-1}b_0)}{z - z_0} + a_{-1}b_1 + a_0b_0 + (a_1b_0 + z_0b_1)(z - z_0) + \cdots \end{aligned}$$

  - Therefore, $\mathrm{Res}_{z_0}(fg) = a_{-1}b_0 = (\mathrm{Res}_{z_0}(f))g(z_0)$

Computation of Residues (Part 2)

- If $f(z)$ is holomorphic at z_0, $f(z_0) = 0$, and $f'(z_0) \neq 0$, then

$$\mathrm{Res}_{z_0}\left(\frac{1}{f(z)}\right) = \frac{1}{f'(z_0)}$$

  - $f(z) = a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$ and $a_1 \neq 0$
  - Therefore,

$$\begin{aligned}\frac{1}{f(z)} &= \frac{1}{a_1(z - z_0) + a_2(z - z_0)^2 + \cdots} \\ &= \left(\frac{a_1^{-1}}{z - z_0}\right)\left(\frac{1}{1 + a_2 a_1^{-1}(z - z_0) + \cdots}\right) \quad = f(z)g(z)\end{aligned}$$

  - By previous result,

$$\mathrm{Res}_{z_0}\left(\frac{1}{f(z)}\right) = \mathrm{Res}_{z_0}(f)g(z_0) = a_1^{-1} = \frac{1}{f'(z_0)}$$

Evaluation of Real Integrals

- Recall the definition of improper integrals:

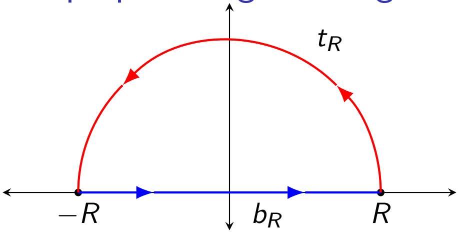
$$\int_{x=a}^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_{x=a}^{x=b} f(x)\,dx$$
$$\int_{x=-\infty}^{x=b} f(x)\,dx = \lim_{a\to-\infty} \int_{x=a}^{x=b} f(x)\,dx$$

- The definition of a two-sided indefinite integral is

$$\int_{x=-\infty}^{x=\infty} f(x)\,dx = \int_{x=-\infty}^{x=0} f(x)\,dx + \int_{x=0}^{x=\infty} f(x)\,dx$$

- Some (but not all) such integrals can be computed using contour integrals

Computation of Improper Integral Using Contour Integral



- Suppose we want to compute $\int_{-\infty}^{\infty} f(x)\, dx$
- Suppose f be extended to a meromorphic function on the upper half-plane
- Let $c_R = (b_R, t_R)$ be the closed contour shown above
- If

  $$\lim_{R\to\infty} \int_{t_R} f(z)\, dz = 0,$$

  then

  $$\lim_{R\to\infty} \int_{c_R} f(z)\, dz = \lim_{R\to\infty} \int_{b_R} f(z)\, dz + \lim_{R\to\infty} \int_{u_R} f(z)\, dz$$

  $$\int^{x=\infty}$$

Computation of Improper Integral Using Contour Integral

- If
$$\lim_{R\to\infty}\int_{t_R} f(z)\,dz = 0,$$
then
$$\lim_{R\to\infty}\int_{c_R} f(z)\,dz = \lim_{R\to\infty}\int_{b_R} f(z)\,dz + \lim_{R\to\infty}\int_{u_R} f(z)\,dz$$
$$= \int_{x=-\infty}^{x=\infty} f(x)\,dx$$
- On the other hand, if f has finitely many poles $z_1,\ldots,z_N$ in the open upper half-plane, then for sufficiently large R,
$$\int_{c_R} f(z)\,dz = 2\pi i\sum_{k=1}^{N}\operatorname{Res}_{z_k} f$$
- It follows that
$$\int_{x=-\infty}^{x=\infty} f(x)\,dx = \lim_{R\to\infty}\int_{c_R} f(z)\,dz = 2\pi i\sum_{k=1}^{N}\operatorname{Res}_{z_k} f$$

Example (Part 1)

- Consider the integral $\displaystyle\int_{x=-\infty}^{x=\infty} \frac{dx}{1+x^4}$
- Let $f(z) = \dfrac{1}{1+z^4}$
- It follows that if $z = Re^{it}$, then $dz = iRe^{it}\,dt$ and therefore

$$\left|\int_{u_R} \frac{dz}{1+z^4}\right| = \left|\int_{t=0}^{t=\pi} \frac{iRe^{it}}{1+R^4e^{i4t}}\,dt\right| \leq \int_{t=0}^{t=\pi} \frac{R}{R^4-R}\,dt \leq \frac{\pi}{R^3-1}$$

- It follows that

$$\lim_{R\to\infty}\left|\int_{u_R} \frac{dz}{1+z^4}\right| = \lim_{R\to\infty} \frac{\pi}{R^3-1} = 0$$

Example (Part 3)

- The poles of f are
$$e^{\frac{i\pi}{4}} = \frac{1+i}{\sqrt{2}}, e^{\frac{i3\pi}{4}} = \frac{-1+i}{\sqrt{2}}, e^{\frac{i5\pi}{4}} = \frac{-1-i}{\sqrt{2}}, e^{\frac{i7\pi}{4}} = \frac{1-i}{\sqrt{2}}$$
- The ones in the upper half-plane are
$$e^{\frac{i\pi}{4}} = \frac{1+i}{\sqrt{2}}, e^{\frac{i3\pi}{4}} = \frac{-1+i}{\sqrt{2}}$$
- By the earlier result,
$$\mathrm{Res}_{z_0}\left(\frac{1}{1+z^4}\right) = \frac{1}{4z_0^3}$$
- Therefore,
$$\int_{x=-\infty}^{x=\infty} \frac{dx}{1+x^4} = \lim_{R\to\infty} \int_{C_R} \frac{dz}{1+z^4} = 2\pi i\left(\frac{1}{4e^{\frac{i3\pi}{4}}} + \frac{1}{4e^{\frac{i9\pi}{4}}}\right)$$
$$= \pi i(e^{\frac{-i3\pi}{4}} + e^{\frac{-i9\pi}{4}}) = \pi i\left(\frac{-1-i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}}\right)$$
$$= \pi\sqrt{2}$$