MATH-GA2450 Complex Analysis

Laurent Series

Deane Yang

Courant Institute of Mathematical Sciences
New York University

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Useful Corollary of Cauchy's Theorem

- Let $O \subset \mathbb{C}$ be open and c be a closed chain in O that is homologous to 0 in O
- Let $z_1, \ldots, z_n \in U$ be distinct points
- Let $\overline{D(z_1, r_1)}, \ldots, \overline{D(z_n, r_n)}$ be disjoint disks and for each $1 \le j \le n$, $c_1 = \partial D(z_j, r_j)$ oriented counterclockwise
  - For example, let
    $$r = \frac{1}{3}\min\{|z_i - z_j| : \ 1 \le i, j \le n\}$$
    and $r_1 = \cdots = r_n = r$
- Let $w_i = W(c, z_i)$
- Then the following hold:
  - In $O \backslash \{z_1, \ldots, z_n\}$, c is homologous to $w_1 c_1 + \cdots + w_n c_n$
  - Given any holomorphic function $f : O \backslash \{z_1, \ldots, z_n\} \to \mathbb{C}$,
    $$\int_c f(z)\, dz = \sum_{j=1}^{n} w_j \int_{c_j} f(z)\, dz$$

Example (Part 1)

- We want to calculate

$$\int_c \frac{dz}{z^4-1} = \int_c \frac{dz}{(z-i)(z+1)(z+i)(z-1)}\, dz$$
$$= \int_c \frac{dz}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)},$$

  where $c = \partial D(0,2)$ and $z_k = i^k$
- For eachh $k = 1,2,3,4$, let $c_k = \partial D(z_k, 1)$, and

$$f_k(z) = \frac{z-z_k}{z^4-1},$$

  which is holomorphic on $D(z_k, \sqrt{2})$

Example (Part 2)

- Since $c - (c_1 + c_2 + c_3 + c_4)$ is null homologous, it follows by the Cauchy integral formula that

$$
\begin{aligned}
&\int_c \frac{dz}{z^4 - 1} \\
&= \sum_{k=1}^{4} \int_{c_k} \frac{f_k(z)}{z - z_k}\, dz \\
&= \sum_{k=1}^{4} 2\pi i f_k(z_k) \\
&= 2\pi i \left(\frac{1}{(i+1)(2i)(i-1)} + \frac{1}{(-1-i)(-1+i)(-1-1)} \right. \\
&\quad \left. + \frac{1}{(-i-i)(-i+1)(-i-1)} + \frac{1}{(1-i)(1-(-1))(1+i)} \right) \\
&= 0
\end{aligned}
$$

Extending a Continuous Function to a Holomorphic Function

- Let $O \subset \mathbb{C}$ be open and

$$c : [a, b] \to \mathbb{C}$$

  be a closed curve
- Given a continuous function $g : c([a, b]) \to \mathbb{C}$, let $f : O \backslash c([a, b]) \to \mathbb{C}$ be the function where for each $z \notin c$,

$$f(z) = \frac{1}{2\pi i} \int_c \frac{g(w)}{w - z}\, dz = \frac{1}{2\pi i} \int_{t=a}^{t=b} \frac{g(c(t))}{c(t) - z} c'(t)\, dt$$

- Then f is holomorphic on $O \backslash c([a, b])$ and for each $z \in O \backslash c([a, b])$ and $k \geq 0$,

$$\frac{f^{(k)}(z)}{k!} = \frac{1}{2\pi i} \int_c \frac{g(w)}{w - z}\, dz$$

Proof

- For each $z \in O \backslash c([a, b])$, there exists $r > 0$ such that
$$\overline{D(z, r)} \subset O \backslash c([a, b])$$
- Since $c([a, b]) \subset \mathbb{C}$, $\overline{D(z, r)}$ are compact, and the function
$$\frac{g(c(t))}{c(t) - z} c'(t)$$
is differentiable with respect to z, it follows that the function
$$f(z) = \frac{1}{2\pi} \int_{t=a}^{t=b} \frac{g(c(t))}{c(t) - z} c'(t)\, dt$$
is differentiable with respect to z
- Differentiating k times with respect to z, we get the formula for $f^{(k)}(z)$

Example

- Let $c(t) = e^{it}$, $0 \leq t \leq 2\pi$
- Observe that

$$\mathbb{C} \backslash c([0, 2\pi]) = D(0,1) \cup \mathbb{C} \backslash \overline{D(0,1)}$$

- Let g be the constant function 1
- If $z \in D(0,1)$, then

$$f(z) = \int_c \frac{dw}{w - z} = W(c, z) = 1$$

- If $z \in \mathbb{C} \backslash \overline{D(0,1)}$, then

$$f(z) = \int_c \frac{dw}{w - z} = W(c, z) = 0$$

- Therefore,

$$f(z) = \begin{cases} 1 & \text{if } |z| < 1 \\ 0 & \text{if } |z| > 1 \end{cases}$$

Another Example

- Let c be a parameterization of $\partial D(0,1)$
- Let $g : \mathbb{C} \to \mathbb{C}$ be holomorphic and let

$$f(z) = \int_c \frac{g(w)}{w - z}\, dw$$

- By the Cauchy integral formula,

$$\begin{aligned} f(z) &= W(c,z)g(z) \\ &= \begin{cases} g(z) & \text{if } |z| < 1 \\ 0 & \text{if } |z| > 1 \end{cases} \end{aligned}$$

Laurent Series

- Given $0 < r < R$, let

$$A(r, R) = \{z \in \mathbb{C} : \; r < |z| < R\}$$

- Let f be a holomorphic function on $A(r, R)$
- Then given $0 < r < s < S < R$, there exists a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

that converges absolutely and uniformly on $\overline{A(s, S)}$
- Moreover,

$$a_n = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{C_S} \dfrac{f(w)}{w^{n+1}}\, dw & \text{if } n \geq 0 \\ \dfrac{1}{2\pi i} \displaystyle\int_{C_s} \dfrac{f(w)}{w^{n+1}}\, dw & \text{if } n < 0 \end{cases}$$

Proof (Part 1)

- Since $c_S - c_s$ is null homologous, by the Cauchy integral formula, if $z \in A(s, S)$, then

$$f(z) = \frac{1}{2\pi i} \int_{c_S - c_s} \frac{f(w)}{w - z}\, dw$$
$$= \frac{1}{2\pi i} \int_{c_S} \frac{f(w)}{w - z}\, dw - \frac{1}{2\pi i} \int_{c_S} \frac{f(w)}{w - z}\, dw$$

- Then

$$\int_{c_S} \frac{f(w)}{w - z}\, dw = \int_{c_S} \frac{f(w)}{w} \left(\frac{1}{1 - \frac{z}{w}} \right) dw$$
$$= \int_{c_S} \frac{f(w)}{w} \sum_{k=0}^{\infty} \left(\frac{z}{w} \right)^k dw$$
$$= \sum_{k=0}^{\infty} \left(\int_{c_S} \frac{f(w)}{w^{k+1}}\, dw \right) z^k$$

- Since $\left| \frac{z}{w} \right| = \frac{|z|}{S} < 1$, the series converges absolutely

Proof (Part 2)

- Since $\left|\frac{w}{z}\right| = \frac{s}{|z|} < 1$, the geometric series

$$\frac{1}{1-\frac{w}{z}} = \sum_{j=0}^{\infty} \left(\frac{w}{z}\right)^j$$

  converges absolutely
- Therefore,

$$\begin{aligned}
\int_{C_s} \frac{f(w)}{w-z}\, dw &= \int_{C_s} \frac{f(w)}{z} \left(\frac{-1}{1-\frac{w}{z}}\right) dw \\
&= -\int_{C_s} \frac{f(w)}{z} \sum_{j=0}^{\infty} \left(\frac{w}{z}\right)^j dw \\
&= \sum_{j=0}^{\infty} \left(\int_{C_s} \frac{f(w)}{w^{-j}}\, dw\right) z^{-j-1} \\
&= \sum_{k=-\infty}^{-1} \left(\int_{C_s} \frac{f(w)}{w^{k+1}}\, dw\right) z^k
\end{aligned}$$

Proof (Part 3)

- It follows that if $z \in A(s, S)$, then

$$f(z) = \sum_{k=-\infty}^{-1} \left(\frac{1}{2\pi i} \int_{c_s} \frac{f(w)}{w^{k+1}}\, dw \right) z^k - \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_{c_S} \frac{f(w)}{w^{k+1}}\, dw \right) z^k$$

- Conversely, if $s < t < S$ and

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k,$$

converges absolutely when $|z| = t$, then for any $n \in \mathbb{Z}$ and

$$\begin{aligned} \frac{1}{2\pi i} \int_{c_t} \frac{f(z)}{z^{n+1}}\, dz &= \frac{1}{2\pi i} \int_{c_t} \sum_{k=-\infty}^{k=\infty} \frac{a_k}{z^{n+1-k}}\, dz \\ &= \sum_{k=-\infty}^{k=\infty} \frac{1}{2\pi i} \int_{c_t} \frac{a_k}{z^{n+1-k}}\, dz \\ &= a_n \end{aligned}$$