MATH-GA2450 Complex Analysis

Winding Number of Oriented Closed Curve
Homologous Closed Curves
Chains
Homologous Forms of Cauchy's Theorem and Cauchy Integral Formula

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Winding Numbers of Homotopic Closed Curves Are Equal

- If $c_0 : [a, b] \to O$ and $c_1 : [a, b] \to O$ are homotopic closed curves in O and $z_0 \notin O$, then

$$W(c_0, z_0) = W(c_1, z_0)$$

- This follows directly from the homotopic form of Cauchy's theorem
- Corollary: If $c : [a, b] \to O$ is a closed curve null homotopic in O, then for any $z_0 \notin O$,

$$W(c, z_0) = 0$$

- Contrapositive: If $c : [a, b] \to O$ is a closed curve and there exists $z_0 \notin O$ such that

$$W(c, z_0) \neq 0,$$

then c is not null homotopic in O

Winding Number Is an Integer (Part 1)

- Let $c : [a, b] \to \mathbb{C}$ be a closed curve and $z_0 \notin c$
- There exists $r : [a, b] \to (0, \infty)$ and $\theta : [a, b] \to \mathbb{R}$ such that

$$c(t) = z_0 + r(t)e^{i\theta(t)},$$

  where $r(b) = r(a)$ and $e^{i\theta(b)} = e^{i\theta(a)}$
- There exists $k \in \mathbb{Z}$ such that $\theta(b) - \theta(a) = 2\pi k$

Winding Number Is an Integer (Part 2)

- The winding number of c around z_0 is

$$\begin{aligned}
W(c, z_0) &= \frac{1}{2\pi i} \int_c \frac{dz}{z - z_0} \\
&= \frac{1}{2\pi i} \int_{t=a}^{t=b} \frac{c'(t)}{c(t) - z_0}\, dt \\
&= \frac{1}{2\pi i} \int_{t=a}^{t=b} \frac{r' e^{i\theta} + i r e^{i\theta}}{r e^{i\theta}}\, dt \\
&= \frac{1}{2\pi i} \int_{t=a}^{t=b} \frac{r'}{r} + i\theta'\, dt \\
&= \frac{1}{2\pi i} (\log(r(b)) - \log(r(a)) + i(\theta(b) - \theta(a))) \\
&= k,
\end{aligned}$$

Homologous Curves

- Goal: Distinguish between an open set with holes from one without holes
- Let $U \subset \mathbb{C}$ be open
- Let $z_0 \in \mathbb{C} \backslash U$
- Idea: z_0 lies in a hole of U if there exists a closed curve c in U that goes around z_0
  - I.e., $W(c, z_0) = 0$
- A closed curve $c : [a, b] \to U$ is **null homologous** in U if for any $z_0 \notin U$,

$$W(c, z_0) = 0$$

- Two closed curves

$$c_1 : [a_1, b_1] \to U, \ c_2 : [a_2, b_2] \to U$$

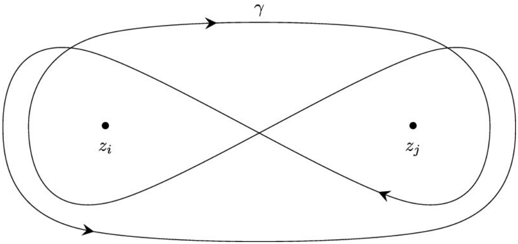
  are **homologous** if for any $z_0 \notin U$,

$$W(c_1, z_0) = W(c_2, z_0)$$

Homotopic $\implies$ Homologous

- If $c_1, c_2 \subset U$ are closed curves that are homotopic in U, then they are homologous in U
  - This follows by the observation above that two closed curves have the same winding number around a point that does not lie on either curve
- Equivalently, if c is a closed curve that is null homotopic in U, then it is homologous to 0 in U
- Converse is not true: If c is a closed curve that is null homologous in U, it is not necessarily null homotopic

Pochhammer Contour



- Null homologous in $\mathbb{C} \backslash \{z_i, z_j\}$, because

$$W(\gamma, z_i) = W(\gamma, z_j) = 0$$

- Not null homotopic
  - Intuition: Any homotopy from c to a point must pass through z_1 or z_2
  - Rigorous proof not so easy

Chains

- A **chain** in $U \in \mathbb{C}$ is the union of a finite set of oriented curves

$$c_1, \dots, c_N$$

in U, where the same curve can appear more than once
- A chain is denoted

$$c = c_1 + \cdots + c_N$$

- If there are m_1 copies of c_1, m_2 copies of c_2, $\dots$, m_n copies of c_n, then the corresponding chain can be written as

$$m_1 c_1 + \cdots + m_n c_n$$

- If $m < 0$, then $mc = (-m)(-c)$
- If $m, n \in \mathbb{Z}$, then

$$mc + nc = (m + n)c$$

- A chain is **closed** if all of the curves are closed

Contour Integral and Winding Number of Chain

- The contour integral of a holomorphic function $f : U \to \mathbb{C}$ on a chain
$$c = m_1 c_1 + \cdots + m_n c_n$$
is defined to be
$$\int_c f(z)\,dz = m_1 \int_{c_1} f(z)\,dz + \cdots + m_n \int_{c_n} f(z)\,dz$$
- If $z_0 \notin c$, then
$$\begin{aligned} W(c, z_0) &= \int_c \frac{dz}{z - z_0} \\ &= m_1 \int_{c_1} \frac{dz}{z - z_0} + \cdots + m_n \int_{c_n} \frac{dz}{z - z_0} \\ &= m_1 W(c_1, z_0) + \cdots + m_n W(c_n, z_0) \end{aligned}$$
- A chain $c \subset U$ is **homologous to** 0 **in** U if for any $z_0 \notin U$,
$$W(c, z_0) = 0$$

Homologous Form of Cauchy's Theorem

- **Theorem.** If $U \subset \mathbb{C}$ is open and $c \subset U$ is a closed chain homologous to 0 in U, then for any holomorphic $f : U \to \mathbb{C}$,

$$\int_c f(z)\,dz = 0$$

- **Corollary.** If c_1, c_2 are homologous closed chains in U, then for any holomorphic $f : U \to \mathbb{C}$,

$$\int_{c_1} f(z)\,dz = \int_{c_2} f(z)\,dz$$

- **Proof of Corollary.** If c_1, c_2 homologous in U, then the closed chain $c = c_2 - c_1$ is homologous to 0 in U and therefore, by the theorem above,

$$0 = \int_c f(z)\,dz = \int_{c_2} f(z)\,dz - \int_{c_1} f(z)\,dz$$

Homologous Form of Cauchy Integral Formula

- **Theorem.** If $U \subset \mathbb{C}$ is open and $c \subset U$ is a closed chain homologous to 0 in U, then for any $z_0 \notin U$ and holomorphic $f : U \to \mathbb{C}$,

$$\frac{1}{2\pi i} \int_c \frac{f(z)}{z - z_0}\, dz = W(c, z_0) f(z_0)$$

- Proofs of the homogous forms of Cauchy's Theorem and Cauchy integral formula are given in book
- If the chain consists of disjoint closed curves, then it follows from Green's theorem