MATH-GA2450 Complex Analysis

Contour Integrals

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Integral of z^n Along Unit Circle

- Let $c : [0, 2\pi] \to \mathbb{C}$ be a parameterization of the unit circle given by
$$c(t) = e^{it}$$
- Given $n \in Z$, by the definition of the integral along a C^1 curve,
$$\begin{aligned}\int_c z^n \, dz &= \int_{t=0}^{t=2\pi} (c(t))^n c'(t) \, dt \\ &= \int_{t=0}^{t=2\pi} (e^{it})^n ie^{it} \, dt \\ &= \int_{t=0}^{t=2\pi} ie^{i(n+1)t} \, dt\end{aligned}$$

Integral of z^n Along Unit Circle for $n \neq -1$ (Part 1)

- If $n \neq -1$, then

$$\frac{d}{dt}\frac{e^{i(n+1)t}}{n+1} = ie^{i(n+1)t}$$

and therefoe by the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_c z^n\,dz &= \int_{t=0}^{t=2\pi} ie^{i(n+1)t}\,dt \\ &= \left.\frac{e^{i(n+1)t}}{n+1}\right|_{t=0}^{t=2\pi} \\ &= \frac{e^{i(n+1)2\pi} - e^0}{n+1} \\ &= 0\end{aligned}$$

Integral of z^n Along Unit Circle for $n \neq -1$ (Part 2)

- If $n \neq -1$, then

$$\frac{d}{dz}\left(\frac{z^{n+1}}{n+1}\right) = z^n$$

and therefore by the Fundamental Theorem of Calculus

$$\begin{aligned}
\int_c z^n \, dz &= \left.\frac{z^{n+1}}{n+1}\right|_{c(0)}^{c(2\pi)} \\
&= \frac{(c(2\pi))^{n+1}}{n+1} - \frac{(c(0))^{n+1}}{n+1} \quad = \frac{e^{i2(n+1)}}{n+1} - \frac{e^0}{n+1} \\
&= 0
\end{aligned}$$

Integral of z^{-1} Along Unit Circle

- $\frac{d}{dt}e^{it} = ie^{it}$
- By the definition of the integral along the path c,

$$\int_c \frac{1}{z}\,dz = \int_{t=0}^{t=2\pi} \frac{1}{c(t)}c'(t)\,dt$$
$$= \int_{t=0}^{t=2\pi} \frac{1}{e^{it}}ie^{it}\,dt$$
$$= i\int_{t=0}^{t=2\pi} 1\,dt$$
$$= 2\pi i$$

- This shows that $\frac{1}{z}$ has no antiderivative on $\mathbb{C}\backslash\{0\}$

Contour Integral Along Oriented Curve

- By definition, if $F : [a, b] \to \mathbb{C}$ is continuous, then

$$\int_{t=b}^{t=a} F(t)\, dt = -\int_{t=a}^{t=b} F(t)\, dt$$

- Given a continuous curve $c : [a, b] \to \mathbb{C}$, define

$$\begin{aligned} -c : [b, a] &\to \mathbb{C} \\ t &\mapsto c(t) \end{aligned}$$

- Given $c : [0, 1] \to \mathbb{C}$, $-c : [1, 0] \to \mathbb{C}$ can also be parameterized by

$$\begin{aligned} \tilde{c} : [0, 1] &\to \mathbb{C} \\ t &\mapsto c(1 - t) \end{aligned}$$

Converse To Fundamental Theorem of Calculus (Part 1)

- Let $O \subset \mathbb{C}$ be nonempty, open, and connected
- Let $f : O \to \mathbb{C}$ be a continuous function such that for any closed piecewise C^1 curve $c : [a, b] \to O$,

$$\int_c f(z)\, dz = 0$$

- Define $F : O \to \mathbb{C}$ as follows:
- Fix $z_0 \in O$
- For each $z \in O$, let $c : [0, 1] \to O$ be a piecewise C^1 curve such that $c(0) = z_0$ and $c(1) = z$
- Define

$$F(z) = \int_c f(z)\, dz$$

Converse To Fundamental Theorem of Calculus (Part 2)

- Let $c_1 : [0,1] \to O$, $c_2 : [0,1] \to \mathbb{C}$ be piecewise C^1 curves such that

$$c_1(0) = c_2(0) = z_0 \text{ and } c_1(1) = c_2(1) = z$$

- The curve $C : [0,2] \to \mathbb{C}$ given by

$$C(t) = \begin{cases} c_1(t) & \text{if } 0 \leq t \leq 1 \\ c_2(2-t) & \text{if } 1 \leq t \leq 2 \end{cases}$$

  is a closed piecewise C^1 curve
- Therefore, by assumption,

$$0 = \int_C f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{-c_2} f(z)\,dz = \int_{c_1} f(z)\,dz - \int_{c_2} f(z)$$

- It follows that the definition of $F(z)$ does not depend on the curve used

Converse To Fundamental Theorem of Calculus (Part 3)

- If $z \in O$, then there exists $\delta > 0$ such that $D(z, \delta) \subset O$
- Given $h \in D(0, \delta)$, consider the curve

$$\begin{aligned} c_h : [0,1] &\to \mathbb{C} \\ t &\mapsto z + th \end{aligned}$$

- For each $t \in [0,1]$,

$$|c_h(t) - z| = |z + th - z| = t|h| < \delta$$

  and therefore, $c_h(t) \in D(z, \delta) \subset O$
- $c_h(0) = z$ and $c_h(1) = z + h$

Converse To Fundamental Theorem of Calculus (Part 4)

- Therefore,

$$\begin{aligned}\frac{F(z+h)-F(z)}{h} - f(z) &= \frac{1}{h}\int_{c_h} f(w)\,dw - f(z) \\ &= \frac{1}{h}\int_{t=0}^{t=1} f(c_h(t))c_h'(t)\,dt - f(z) \\ &= \frac{1}{h}\int_{t=0}^{t=1} f(z+th)h\,dt - \int_{t=0}^{t=1} f(z)\,dt \\ &= \int_{t=0}^{t=1} f(z+th) - f(z)\,dt\end{aligned}$$

Converse To Fundamental Theorem of Calculus (Part 4)

- Since f is continuous, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$z + h \in D(z, \delta) \implies |f(z+h) - f(z)| < \epsilon$$

- It follows that if $|h| < \delta$,

$$\begin{aligned}\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| &= \left|\int_{t=0}^{t=1} f(z+th) - f(z)\, dt\right| \\ &\leq \int_{t=0}^{t=1} |f(z+th) - f(z)|\, dt \\ &\leq \epsilon\end{aligned}$$

- It follows that

$$F'(z) = \lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z)$$