MATH-GA2450 Complex Analysis

Contour Integral
Fundamental Theorem of Calculus

Deane Yang

Courant Institute of Mathematical Sciences
New York University

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Piecewise C^1 curves

- A map $c : [a, b] \to \mathbb{C}$ can be written as

$$c(t) = x(t) + iy(t), \;\; a \leq t \leq b,$$

  where a, b are real-valued functions
- The map c is a C^1 **curve** if the real functions

$$x : [a, b] \to \mathbb{R} \text{ and } b : [a, b] \to \mathbb{R}$$

  are differentiable and their derivatives are continuous
- A map $c : [a, b] \to \mathbb{C}$ is a **piecewise** C^1 **curve** or **path** if there exists a partition of $[a, b]$,

$$a = t_0 < t_1 < \cdots < t_N = b$$

  such that for each $1 \leq k \leq N$, the map c restricted to $[t_{k-1}, t_k]$,

$$c : [t_{k-1}, t_k] \to \mathbb{C} \text{ is a } C^1 \text{ curve}$$

Connected Open Domain

- An open $O \subset \mathbb{C}$ is **connected** if for any $z_0, z_1 \in O$, there exists a piecewise C^1 curve

$$c : [t_0, t_1] \to O$$

such that

$$c(t_0) = z_0 \text{ and } c(t_1) = z_1$$

Derivative of C^1 curve

- If $c = x + iy : [a, b] \to \mathbb{C}$ is a C^1 curve, then for each $t \in (a, b)$,

$$\begin{aligned} c'(t) &= \lim_{\delta \to 0} \frac{c(t+\delta) - c(t)}{\delta} \\ &= \lim_{\delta \to 0} \frac{(x(t+\delta) + iy(t+\delta)) - (x(t) + iy(t))}{\delta} \\ &= \lim_{\delta \to 0} \frac{x(t+\delta) - x(t)}{\delta} + i \lim_{\delta \to 0} \frac{y(t+\delta) - y(t)}{\delta} \\ &= x'(t) + iy'(t) \end{aligned}$$

Rules of Differentiation

- If $F : [a, b] \to \mathbb{C}$ and $G : [a, b] \to \mathbb{C}$ are C^1 curves, the

$$(F + G)' = F' + G'$$
$$(FG)' = F'G + FG'$$

- If $G(t) \neq 0$ for all $t \in [a, b]$, then

$$\left(\frac{F}{G}\right)' = \frac{F'G - FG'}{G^2}$$

Chain Rule (Part 1)

- We want to show that if $f : O \to \mathbb{R}$ is holomorphic and $c : [a, b] \to O$ is C^1, then

$$(f \circ c)'(t) = f'(c(t))c'(t)$$

- For each $t \in (a, b)$, let $(t_k \in (a, b) : \ k \geq 0)$ be a sequence such that

$$\lim_{k \to \infty} t_k = t$$

and, for each $k \geq 1$,

$$c(t_k) \neq c(t_{k-1})$$

(If no such sequence exists, then c is constant in an open interval containing t and therefore

$$(f \circ c)'(t) = 0$$

Chain Rule (Part 1)

- By the definition of the derivative,

$$\begin{aligned}(f \circ c)'(t) &= \lim_{\delta \to 0} \frac{f(c(t+\delta)) - f(c(t))}{\delta} \\ &= \lim_{\delta \to 0} \frac{f(c(t+\delta)) - f(c(t))}{c(t+\delta) - c(t)} \frac{c(t+\delta) - c(t)}{\delta} \\ &= \left(\lim_{\delta \to 0} \frac{f(c(t+\delta)) - f(c(t))}{c(t+\delta) - c(t)} \right) \left(\lim_{\delta \to 0} \frac{c(t+\delta) - c(t)}{\delta} \right) \\ &= f'(c(t))c'(t)\end{aligned}$$

Integral of Complex-Valued Function of One Real Variable

- Let $F = u + iv : [a, b] \to \mathbb{C}$ be continuous
- The integral of F over the interval $[a, b]$ is defined to be

$$
\begin{aligned}
\int_{t=a}^{t=b} F(t)\, dt &= \int_{t=a}^{t=b} u(t) + vi(t)\, dt \\
&= \int_{t=a}^{t=b} u(t)\, dt + i \int_{t=a}^{t=b} v(t)\, dt
\end{aligned}
$$

Fundamental Theorem of Calculus

If $F : [a, b] \to \mathbb{C}$ is C^1, then

$$\begin{aligned}
\int_{t=a}^{t=b} F'(t)\,dt &= \int_{t=a}^{t=b} u'(t) + iv'(t)\,dt \\
&= \int_{t=a}^{t=b} u'(t)\,dt + i\int_{t=a}^{t=b} v'(t)\,dt \\
&= u(b) - u(a) + i(v(b) - v(a)) \\
&= u(b) + iv(b) - (u(a) + iv(a)) \\
&= F(b) - F(a)
\end{aligned}$$

Integration by Parts

If $F : [a, b] \to \mathbb{C}$ and $G : [a, b] \to \mathbb{C}$ are C^1, then

$$\begin{aligned} F(b)G(b) - F(a)G(a) &= \int_{t=a}^{t=b} (FG)'(t)\,dt \\ &= \int_{t=a}^{t=b} F'(t)G(t) + F(t)G'(t)\,dt \\ &= \int_{t=a}^{t=b} F'(t)G(t)\,dt + \int_{t=a}^{t=b} F(t)G'(t)\,dt \end{aligned}$$

Fundamental Theorem of Calculus on Piecewise C^1 Curve

If $F : [a, b] \to \mathbb{C}$ is a piecewise C^1 curve, then

$$\int_{t=a}^{t=b} F'(t)\, dt = F(b) - F(a)$$

because if $a = t_0 < t_1 < \cdots < t_N = b$, then

$$\begin{aligned}
&\int_{t=a}^{t=b} F'(t)\, dt \\
&= \int_{t=t_0}^{t=t_1} F'(t)\, dt + \cdots + \int_{t_{N-1}}^{t_N} F'(t)\, dt \\
&= F(t_1) - F(t_0) + F(t_2) - F(t_1) + \cdots + F(t_N) - F(t_{N-1}) \\
&= F(t_N) - F(t_0) \\
&= F(b) - F(a)
\end{aligned}$$

Contour Integral of Function Along Piecewise C^1 Curve

- Let $O \subset \mathbb{C}$ be open and $f : O \to \mathbb{C}$ be continuous
- Given a piecewise C^1 curve $c : [a, b] \to O$, the integral of f along c is defined to be

$$\int_c f(z)\, dz = \int_{t=a}^{t^b} f(c(t))c'(t)\, dt$$

Fundamental Theorem of Calculus

If $f : O \to \mathbb{C}$ is holomorphic, then for any piecewise C^1 curve $c : [a, b] \to O$,

$$\begin{aligned}\int_c f'(z)\,dz &= \int_{t=a}^{t=b} f'(c(t))c'(t)\,dt \\ &= \int_{t=a}^{t=b} (f \circ c)'(t)\,dt \\ &= f(c(b)) - f(c(a)) \\ &= f(z_{\text{end}}) - f(z_{\text{start}})\end{aligned}$$

Contour Integral of Holomorphic Function Along Closed Curve

- A continuous curve $c : [a, b] \to \mathbb{C}$ is **closed** if $c(a) = c(b)$
- If $f : O \to \mathbb{C}$ is holomorphic and $c : [a, b] \to O$ is a closed piecewise C^1 curve, then

$$\int_c f'(z)\, dz = 0$$

- The curve can be written as $z = c(t)$ and therefore $dz = c'(t)\, dt$

Fundamental Theorem of Calculus

- If $f : O \to \mathbb{C}$ is holomorphic, then $F : O \to \mathbb{C}$ is an **antiderivative** or **primitive** of f if

$$F' = f$$

- **Fundamental Theorem of Calculus** If $F : O \to \mathbb{C}$ is an antiderivative of $f : O \to \mathbb{C}$, the for any piecewise C^1 curve $c : [a, b] \to O$,

$$\int_c f(z)\, dz = F(c(b)) - F(c(a))$$

- **Fundamental Theorem of Calculus for Closed Curve** If $f : O \to \mathbb{C}$ has an antiderivative on O, then for any closed piecewise C^1 curve $c : [a, b] \to O$,

$$\int_c f(z)\, dz = 0$$