MATH-GA2450 Complex Analysis

Algebra of Analytic Functions
Binomial Series
Differentiation of Power Series

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Algebra of Analytic Functions

- If f and g are analytic on an open $O \subset \mathbb{C}$ containing z_0, then there exists $r > 0$ such that $f + g$ is analytic on $D(z_0, r)$
- If f and g are analytic on an open $O \subset \mathbb{C}$ containing z_0, then there exists $r > 0$ such that fg is analytic on $D(z_0, r)$
- If f is analytic on an open $O \subset \mathbb{C}$ containing z_0 and $f(z_0) \neq 0$, then there exists $r > 0$ such that $1/f$ is analytic on $D(z_0, r)$
- If f is analytic on an open $O \subset \mathbb{C}$ and g analytic on an open $O' \subset \mathbb{C}$ containing $f(z_0)$, then there exists $r > 0$ such that $g \circ f$ is analytic on $D(z_0, r)$

Binomial Coefficients

- Recall that if $n \in \mathbb{Z}_+$, then for any $r \in \mathbb{R}$,

$$(1+r)^n = \sum_{k=0}^{n} \binom{n}{k} r^k,$$

where

$$\binom{n}{k} = \begin{cases} 1 & \text{if } k = 0 \\ \dfrac{n!}{k!(n-k)!} = \dfrac{n(n-1)\cdots(n-(k-1))}{k(k-1)\cdots 1} & \text{if } k \geq 1 \end{cases}$$

- For any $\alpha \in R$ and positive integer k, define

$$\binom{\alpha}{k} = \begin{cases} 1 & \text{if } k = 0 \\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-(k-1))}{k(k-1)\cdots 1} & \text{if } k \geq 1 \end{cases}$$

Binomial Series

- Given $\alpha \in \mathbb{R}$, the binomial series is defined to be

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} z^k$$

- If $k \geq 1$, then

$$\begin{aligned}
&\left| \left(\binom{\alpha}{k+1} z^{k+1} \right) \left(\binom{\alpha}{k} z^k \right)^{-1} \right| \\
&= \left| \left(\frac{\alpha(\alpha-1)\cdots(\alpha-k)}{(k+1)k(k-1)\cdots 1} \right) \left(\frac{k(k-1)\cdots 1}{\alpha(\alpha-1)\cdots(\alpha-(k-1))} \right) \right| |z| \\
&= \left| \frac{\alpha-k}{k+1} \right| |z|
\end{aligned}$$

- The limit of this as $k \to \infty$ is $|z|$
- Therefore, the binomial series has radius of convergence 1

Differentiation of Power Series

- Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ have radius of convergence $R > 0$ and
$$\lim_{k \to \infty} \frac{|a_{k+1}|}{|a_k|} = \frac{1}{R}$$
- **Theorem:** The power series $\displaystyle\sum_{k=1}^{\infty} k a_k z^{k-1}$ also has radius of convergence R and
$$f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} z^k$$
- Radius of convergence for special case: If
$$\lim_{k \to \infty} \frac{|a_{k+1}|}{|a_k|} = \frac{1}{R},$$
then
$$\lim_{k \to \infty} \frac{|(k+1)a_{k+1}|}{|k a_k|} = \lim_{k \to} \frac{k+1}{k} \frac{|a_{k+1}|}{|a_k|} = \frac{1}{R},$$

Derivative of Analytic Function (Part 1)

- We want to show that

$$\begin{aligned}\sum_{k=1}^{\infty} k a_k z^{k-1} &= f'(z) \\ &= \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\sum_{k=0}^{\infty} a_k (z+h)^k - \sum_{k=0}^{\infty} a_k z^k\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\sum_{k=0}^{\infty} a_k ((z+h)^k - z^k)\right)\end{aligned}$$

Convergence of Series with Nonnegative Terms

- A series

$$\sum_{k=k_0}^{\infty} r_k,$$

where each $r_k \geq 0$, converges if and only if there exists $M > 0$ such that for any $N \geq 0$,

$$\sum_{k=k_0}^{N} r_k \leq M$$

- This follows from the fact that

$$S_N = \sum_{k=0}^{N} r_k$$

is a bounded increasing sequence and therefore converges

Derivative of Analytic Function (Part 2)

- Given z such that $|z| < R$, let h satisfy $|h| < R - |z|$
- For each $k \geq 0$, if

$$Q_k(z,h) = ((z+h)^k - z^k),$$

then

$$\begin{aligned}\sum_{k=0}^{\infty} |a_k Q_k(z,h)| &= \sum_{k=0}^{\infty} |a_k (z+h)^k - a_k z^k| \\ &\leq \sum_{k=0}^{\infty} |a_k (z+h)^k| + |a_k z^k| \\ &= \sum_{k=0}^{\infty} |a_k (z+h)^k| + \sum_{k=0}^{\infty} |a_k z^k| < \infty\end{aligned}$$

- It follows that

$$\sum_{k=0}^{\infty} a_k Q_k(z,h) \text{ converges absolutely}$$

Derivative of Analytic Function (Part 3)

- Observe that

$$Q_0(z,h) = 0$$
$$Q_1(z,h) = z + h - z = h$$

and if $k \geq 2$,

$$\begin{aligned} Q_k(z,h) &= ((z+h)^k - z^k) \\ &= \left(\left(\sum_{j=0}^{k} \binom{k}{j} z^{k-j} h^j \right) - z^k \right) \\ &= \left(kz^{k-1}h + \sum_{j=2}^{k} \binom{k}{j} z^{k-j} h^j \right) \\ &= \left(kz^{k-1}h + h^2 \sum_{j=2}^{k} \binom{k}{j} z^{k-j} h^{j-2} \right) \end{aligned}$$

Derivative of Analytic Function (Part 4)

- If $k \geq 2$, then

$$Q_k(z, h) = kz^{k-1} + h^2 P_k(z, h),$$

where

$$P_k(z, h) = \sum_{j=2}^{k} \binom{k}{j} z^{k-j} h^{j-2}$$

- If $\delta = R - |z|$, then $|h| < \delta$ and therefore

$$|P_k(z, h)| \leq \sum_{j=2}^{k} \binom{k}{j} |z|^{k-j} |h|^{j-2} < \sum_{j=2}^{k} \binom{k}{j} |z|^{k-j} \delta^{j-2} = P_k(|z|, \delta)$$

Derivative of Analytic Function (Part 5)

- It follows that

$$\begin{aligned}
\sum_{k=2}^{\infty} |a_k h^2 P_k(z,h)| &\leq \sum_{k=2}^{\infty} |h|^2 |a_k| P_k(|z|,\delta) \\
&= \sum_{k=2}^{\infty} |a_k| Q_k(|z|,\delta) - k|a_k||z|^{k-1} \\
&\leq \sum_{k=2}^{\infty} |a_k| Q_k(|z|,\delta) + \sum_{k=2}^{\infty} k|a_k||z|^{k-1} \\
&< \infty
\end{aligned}$$

Derivative of Analytic Function (Part 6)

- Putting everything together, we get

$$
\begin{aligned}
&\lim_{h\to 0}\left|\frac{f(z+h)-f(z)}{h}-\sum_{k=1}^{\infty} k a_k z^{k-1}\right| \\
&= \lim_{h\to 0}\left|\frac{1}{h}\sum_{k=0}^{\infty} a_k((z+h)^k - z^k) - k a_k z^{k-1}\right| \\
&= \lim_{h\to 0}\left|\frac{1}{h}\sum_{k=2}^{\infty} a_k h^2 P_k(z,h)\right| \\
&= \lim_{h\to 0}|h|\left|\sum_{k=2}^{\infty} a_k P_k(z,h)\right| \\
&\leq \lim_{h\to 0}|h|\sum_{k=2}^{\infty} |a_k| P_k(|z|,\delta) \\
&= 0
\end{aligned}
$$