MATH-GA2450 Complex Analysis

Algebra of Formal Power Series
Laurent Series

Deane Yang

Courant Institute of Mathematical Sciences
New York University

September 19, 2024

Algebra of Formal Power Series

- Essentially the same as algebra of polynomials
- No claims about convergence are made here
- Consider two power series:

$$\sum_{k=0}^{\infty} a_k z^k \text{ and } \sum_{k=0}^{\infty} b_k z^k$$

- Addition

$$\sum_{k=0}^{\infty} a_k z^k + \sum_{k=0}^{\infty} b_k z^k = \sum_{k=0}^{\infty} (a_k + b_k) z^k$$

Multiplication of Power Series

$$
\begin{aligned}
&\left(\sum_{k=0}^{\infty} a_k z^k\right)\left(\sum_{k=0}^{\infty} b_k z^k\right) \\
&= (a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots)(b_0 + b_1 z + b_2 z + b_3 z^3 + \cdots) \\
&= a_0 b_0 + (a_1 b_0 + a_0 b_1) z + (a_2 b_0 + a_1 b_1 + a_0 b_2) z^2 \\
&\quad + (a_3 b_0 + a_2 b_1 + a_1 b_2 + a_0 b_3) z^3 + \cdots \\
&= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_{k-j} b_j\right) z^k
\end{aligned}
$$

Reciprocal of Power Series (Part 1)

- Given a power series

$$\sum_{k=0}^{\infty} a_k z^k,$$

want to solve for the coefficients of the power series

$$\sum_{k=0}^{\infty} c_k z^k = \frac{1}{\sum_{k=0}^{\infty} a_k z^k}$$

- Use multiplication formula

$$\begin{aligned} 1 &= \left(\sum_{k=0}^{\infty} c_k z^k\right)\left(\sum_{k=0}^{\infty} a_k z^k\right) \\ &= (c_0 + c_1 z + c_2 z + c_3 z^3 + \cdots)(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots) \\ &= c_0 a_0 + (c_1 a_0 + c_0 a_1)z + (c_2 a_0 + c_1 a_1 + c_0 a_2)z^2 \\ &\quad + (c_3 a_0 + c_2 a_1 + c_1 a_2 + c_0 a_3)z^3 + \cdots \end{aligned}$$

Reciprocal of Power Series (Part 1)

- Therefore,

$$\begin{aligned} c_0 a_0 &= 1 \\ c_0 a_1 + c_1 a_0 &= 0 \\ c_0 a_2 + c_1 a_1 + c_2 a_0 &= 0 \\ c_0 a_3 + c_1 a_2 + c_2 a_1 + c_3 a_0 &= 0 \end{aligned}$$

- Solve for c_0, c_1, c_2, c_3:

$$\begin{aligned} c_0 &= a_0^{-1} \\ c_1 &= -a_0^{-1} c_0 a_1 \\ c_2 &= -a_0^{-1}(c_1 a_1 + c_0 a_2) \\ c_3 &= -a_0^{-1}(c_2 a_1 + c_1 a_2 + c_0 a_3) \end{aligned}$$

- Solve for c_k in terms of recursive formula:

$$c_k = -a_0^{-1} \sum_{j=1}^{k} a_{k-j} c_j$$

Division of Power Series (Part 1)

- Given power series

$$\sum_{k=0}^{\infty} a_k z^k \text{ and } \sum_{k=0}^{\infty} b_k z^k,$$

want to solve for the coefficients of the power series

$$\sum_{k=0}^{\infty} c_k z^k = \frac{\sum_{k=0}^{\infty} a_k z^k}{\displaystyle\sum_{k=0}^{\infty} b_k z^k}$$

Division of Power Series (Part 2)

- Use multiplication formula

$$
\begin{aligned}
\sum_{k=0}^{\infty} a_k z^k
&= \left(\sum_{k=0}^{\infty} b_k z^k\right)\left(\sum_{k=0}^{\infty} c_k z^k\right) \\
&= (b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots)(c_0 + c_1 z + c_2 z + c_3 z^3 + \cdots) \\
&= b_0 c_0 + (b_1 c_0 + b_0 c_1) z + (b_2 c_0 + b_1 c_1 + b_0 c_2) z^2 \\
&\quad + (b_3 c_0 + b_2 c_1 + b_1 c_2 + b_0 c_3) z^3 + \cdots \\
&= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} b_{k-j} c_j\right) z^k
\end{aligned}
$$

Division of Power Series (Part 3)

- Therefore,

$$\begin{aligned} b_0 c_0 &= a_0 \\ b_0 c_1 + b_1 c_0 &= a_1 \\ b_0 c_2 + b_1 c_1 + b_2 c_0 &= a_2 \\ b_0 c_3 + b_1 c_2 + b_2 c_1 + b_3 c_0 &= a_3 \end{aligned}$$

- Solve for c_0, c_1, c_2, c_3:

$$\begin{aligned} c_0 &= b_0^{-1} a_0 \\ c_1 &= b_0^{-1}(a_1 - b_1 c_0) \\ c_2 &= b_0^{-1}(a_2 - b_1 c_1 - b_2 c_0) \\ c_3 &= b_0^{-1}(a_3 - b_1 c_2 - b_2 c_1 - b_3 c_0) \end{aligned}$$

- Solve for c_k in terms of recursive formula:

$$c_k = b_0^{-1} \left(a_k - \sum_{j=1}^{k} b_b c_{k-j} \right)$$

Laurent Series

- A **Laurent series** is an infinite sum of the form

$$a_{-m}z^{-m} + \cdots + a_{-1}z^{-1} + a_0 + a_1 z + \cdots = \sum_{k=-m}^{\infty} a_k z^k$$

- Algebraic formulas are essentially the same as for power series
- If $-m < -n$,

$$\sum_{k=-m}^{\infty} a_k z^k + \sum_{k=-n}^{\infty} b_k z^k = \sum_{k=-m}^{-n} a_k z^k + \sum_{k=-n}^{\infty} (a_k + b_k) z^k$$

$$\left(\sum_{k=-m}^{\infty} a_k z^k\right)\left(\sum_{k=-n} b_k z^k\right) = \sum_{k=-m+n}^{\infty} \left(\sum_{j=-m}^{k} a_j b_{k-j}\right) z^k$$

- The **order** of a Lauren series

$$\sum_{k=-m}^{\infty} a_k z^k,$$

where $a_{-m} \neq 0$ is defined to be $-m$

Division of Power Series

- A power series **vanishes to order** n if the first nonzero term is $a_n z^n$
- Example: The power series of $\sin(z)$ vanishes to order 1,

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

- The reciprocal of a power series that vanishes to order greater than 0 is a Laurent series
- The quotient of a power series, where the denominator vanishes to order k, is a Laurent series

Reciprocal of $\sin(z)$

- The Taylor series of $\sin(z)$ is

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

- If

$$\frac{1}{\sin(z)} = c_{-1}z^{-1} + c_0 + c_1 z + \cdots ,$$

then

$$\begin{aligned} 1 &= \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)\left(c_{-1}z^{-1} + c_0 + c_1 z + \cdots\right) \\ &= c_{-1} + c_0 z + \left(c_1 - \frac{c_{-1}}{3!}\right) z^2 + \left(c_2 - \frac{c_0}{3!}\right) z^3 + \cdots \end{aligned}$$

- Therefore,

$$c_{-1} = 1,\ \ c_0 = 0,\ \ c_1 = \frac{1}{6},\ \ c_2 = 0$$

and

$$\frac{1}{\sin(z)} = \frac{1}{z} + \frac{z}{6} + \cdots$$