MATH-GA2450 Complex Analysis

Complex Multiplication is Conformal
Holomorphic Implies Conformal
Series
Power Series

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Geometry of Complex Multiplication

- If $z = re^{i\theta}$ and $w = \rho e^{i\phi}$, then

$$wz = \rho r e^{i(\theta+\phi)}$$

- Multiplying z by w rotates z by the angle ϕ and rescales it by a factor of ρ
- The angle from $z_1 = r_1 e^{i\theta_1}$ to $z_2 = r_2 e^{i\theta_2}$ is α if and only if

$$\frac{z_2}{z_1} = \frac{r_2}{r_1} e^{i(\theta_2 - \theta_1)} = \frac{|z_2|}{|z_1|} e^{i\alpha}$$

- It follows that if the angle from wz_1 to wz_2 is β,

$$\frac{|wz_2|}{|wz_1|} e^{i\beta} = \frac{wz_2}{wz_1} = \frac{z_2}{z_1} = \frac{|z_2|}{|z_1|} e^{i\alpha}$$

- Therefore, the angle from wz_1 to wz_2 is equal to the angle from z_1 to z_2

Composition of Curve by Holomorphic Function

- Let $I \subset \mathbb{R}$ be a nonempty open interval and $c : I \to O$ be a parameterized curve in an open $O \subset \mathbb{C}$
- If $z(t) = x(t) + iy(t)$, then its velocity is

$$c'(t) = x'(t) + iy'(t)$$

- Let $f : O \to \mathbb{C}$ be holomorphic and denote

$$f(x + iy) = u(x, y) + iv(x, y)$$

- Consider the curve

$$(f \circ c) : I \to f(O)$$

Velocity of Composition of Curve by Holomorphic Function

- By the chain rule and the Cauchy-Riemann equations,

$$\begin{aligned}
(f \circ c)'(t) &= \frac{d}{dt}(u(x(t), y(t)) + iv(x(t), y(t))) \\
&= u_x(x(t), y(t))x'(t) + u_y(x(t), y(t))y'(t) \\
&\quad i(v_x(x(t), y(t))x'(t) + v_y(x(t), y(t))y'(t)) \\
&= u_x x' - v_x y' + i(v_x x' + u_x y') \\
&= (u_x + iv_x)x' + (-v_x + iu_x)y' \\
&= (u_x + iv_x)(x' + iy') \\
&= f'(c(t))c'(t)
\end{aligned}$$

- Velocity of the curve $f \circ c$ at $(f \circ c)(t)$ is the velocity of c at $c(t)$ multiplied by $f'(c(t))$

Angle Between Two Intersecting Curves

- Let c_1 and c_2 be parameterized curves in $O \subset \mathbb{C}$
- Suppose the curves cross at $z_0 = c_1(t_0) = c_2(t_0)$ and $c_1'(t_0), c_2'(t_0) \neq 0$
- The angle between the two curves at z_0 is defined to be the angle from $c_1'(t_0)$ to $c_2'(t_0)$
- If α is the angle from $c_1'(t_0)$ to $c_2'(t_0)$, then

$$\frac{c_2'(t_0)}{c_1'(t_0)} = \frac{|c_2'(t_0)|}{|c_1'(t_0)|} e^{i\alpha}$$

Holomorphic Implies Conformal

- Let $f : O \to \mathbb{C}$ be holomophic and suppose $f'(z_0) \neq 0$
- The two curves $f \circ c_1$ and $f \circ c_2$ cross at $f(z_0) = f(c_1(t_0)) = f(c_2(t_0))$
- The angle between these two curvaes at $f(z_0)$ is the angle from

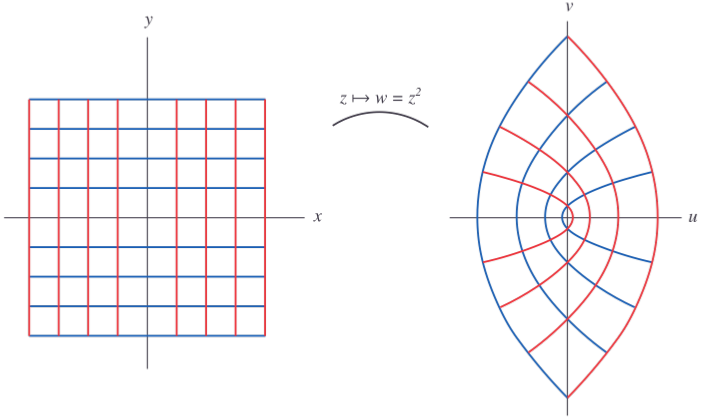
$$(f \circ c_1)'(t_0) = f'(c_1(t_0))c_1'(t_0) = f'(z_0)c_1'(t_0)$$

  to

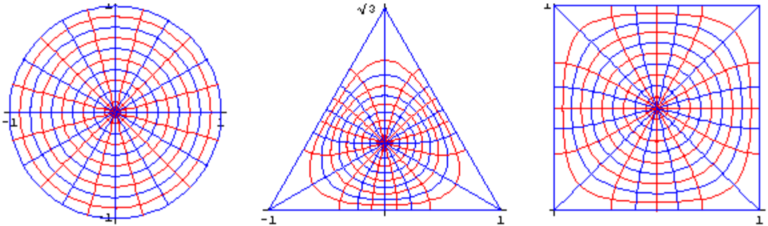
$$(f \circ c_2)'(t_0) = f'(c_2(t_0))c_2'(t_0) = f'(z_0)c_2'(t_0)$$

- This is equal to the angle from $c_1'(t_0)$ to $c_2'(t_0)$
- At each $z \in O$, the holomorphic function $f : O \to \mathbb{C}$ preserves the angle of any two curves passing through z_0
- An angle-preserving map is called a **conformal map**

$z \mapsto z^2$



Examples of Conformal Maps



Series

- A series is an infinite sum of complex numbers

$$\sum_{k=0}^{\infty} z_k = z_0 + z_1 + \cdots$$

- In general, it is just a formula and not a number
- For each $N \geq 0$, the N-th partial sum is

$$S_N = z_0 + \cdots + z_N \in \mathbb{C}$$

- The series **converges** to S if

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{k=0}^{N} z_k = S$$

or equivalently,

$$\lim_{N\to\infty} |S_N - S| = 0$$

- Example:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \text{ converges}$$

Fundamental Example: Geometric Series

- Given $z \in \mathbb{C}$ and $n \in \mathbb{Z}_+$,

$$(1-z)(1+z+\cdots+z^n)$$
$$= (1+z+\cdots+z^n) - (z+z^2+\cdots+z^n+z^{n+1}) = 1 - z^{n+1}$$

- Therefore,

$$\left| \left(\sum_{k=0}^{N} z^k \right) - \frac{1}{1-z} \right| = \left| \frac{z^{N+1}}{1-z} \right| = \frac{|z|^{N+1}}{|1-z|}$$

- If $|z| < 1$, then $\lim_{N \to \infty} \frac{|z|^{N+1}}{|1-z|} = 0$ and therefore

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$$

- If $|z| \geq 1$, then $|z|^N \geq 1$, which implies $(|z|^N : N \in \mathbb{Z}_+)$ does not converge to zero, and therefore the series diverges

Rearrangement of a Series

- A rearrangement of a series

$$\sum_{k=1}^{\infty} z_k$$

is a series

$$\sum_{k=0}^{\infty} z_{\sigma(k)},$$

where $\sigma : \mathbb{Z}_+ \to \mathbb{Z}_+$ is a bijective map ("permutation")

Absolutely Convergent Series

- The series **converges absolutely** to S if

$$\sum_{k=0}^{\infty} |z|_k \text{ converges}$$

- An absolutely convergent series is convergent
- Any rearrangement of an absolutely convergent series is absolutely convergent and has the same limit
- Examples:
  - $\sum_{k=1} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges
  - $\sum_{k=1} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$ converges but not absolutely
  - $\sum_{k=1} \frac{(-1)^{k+1}}{k^2} = 1 - \frac{1}{4} + \frac{1}{9} - \cdots$ converges absolutely

Ratio Test For Absolute Convergence

- Consider the series $\sum_{k=0}^{\infty} z_k$
- Assume every term is nonzero
- Assume that the following limit exists:

$$r = \lim_{k \to \infty} \frac{|z_{k+1}|}{|z_k|}$$

- If $r < 1$, then the series converges absollutely
- $r > 1$, then the series diverges
- If $r = 1$, then the test is inconclusive

Power Series

- A power series centered at 0 is a polynomial of infinite degree,

$$\sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + cdots,$$

  where $a_0, a_1, \ldots$ are coefficients
- A power series centered at z_0 is of the form

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k = a_0 + a_1(z - z_0) + \cdots,$$

  where $a_0, a_1, \ldots$ are coefficients
- In general, this is just a formula and not equal to a function

Radius of Convergence

- Consider

$$\sum_{k=0}^{\infty} a_k z^k$$

- By the ratio test, if

$$\lim_{k\to\infty} \frac{|a_{k+1} z^{k+1}|}{|a_k z^k|} = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} |z| = |z| \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}, < 1$$

  the series converges
- The **radius of convergence** is defined to be

$$R = \left(\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} \right)^{-1},$$

- The power series converges absolutely if $|z| < R$ and diverges if $|z| > R$

Radius of Convergence

- More generally, consider

$$\sum_{j=0}^{\infty} a_j z^{k_j}$$

- By the ratio test, if

$$\lim_{j\to\infty} \frac{|a_{j+1} z^{k_{j+1}}|}{|a_j z^{k_j}|} = \lim_{j\to\infty} \frac{|a_{j+1}|}{|a_j|} |z|^{k_{j+1}-k_j} = |z|^{k'} \lim_{j\to\infty} \frac{|a_{j+1}|}{|a_j|} < 1,$$

  where $k' = \lim_{j\to\infty} (k_{j+1} - k_j)$, the series converges
- The **radius of convergence** is defined to be

$$R = \left(\lim_{j\to\infty} \frac{|a_{j+1}|}{|a_j|} \right)^{\frac{-1}{k'}}$$

- The power series converges absolutely if $|z| < R$ and diverges if $|z| > R$