MATH-GA2450 Complex Analysis

Topology of $\mathbb{C}$
Limit of Sequence
Convergent and Cauchy Sequences

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Length and Distance

- Recall that the **magnitude** or **length** of $z = x + iy \in C$ is

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$$

- The **distance** from z to w is

$$d(z, w) = |w - z| = |z - w|$$

- The **open disk** of radius r cenetered at x is

$$D(z, r) = \{w \in \mathbb{C} : \ d(z, w) < r\} = \{w \in \mathbb{C} : \ |w - z| < r\}$$

- The **closed disk** of radius r cenetered at x is

$$\overline{D}(z, r) = \{w \in \mathbb{C} : \ d(z, w) < r\} = \{w \in \mathbb{C} : \ |w - z| \leq r\}$$

Open and Closed Sets

- $S \subset \mathbb{C}$ is **open** if for each $z \in S$, there is an $r > 0$ such that
$$D(z, r) \subset S$$
- $T \subset \mathbb{C}$ is **closed** if its complement $\mathbb{C} \backslash T$ is open
- Examples: If $r > 0$, then
- $D(z, r)$ is open
- $\overline{D}(z, r)$ is closed
- $\mathbb{C} \backslash D(z, r)$ is closed
- $\mathbb{C} \backslash \overline{D}(z, r)$ is open

Bounday, Closure, Inteior of a Subset

- A point $z \in \mathbb{C}$ is a **boundary point** of $S \subset \mathbb{C}$ if for any $r > 0$,

$$D(z, r) \cap S \neq \emptyset$$

- The set of all boundary points of $S \subset$ is the **boundary** of S, which is denoted by ∂S
- The **closure** of S is

$$\overline{S} = S \cup \partial S$$

  - The smallest closed set containing S
- ∂S and $\overline{S}$ are closed sets
- The **interior** of S is $S \backslash \partial S$

Bounded Sets

- $S \subset \mathbb{C}$ is **bounded** if there exists $r > 0$ such that

$$S \subset D(0, r),$$

  i.e., if for every $z \in S$,

$$|z| < r$$

- Examples
  - For any $z \in \mathbb{C}$ and $r > 0$, $D(z, r)$ is bounded
  - The **open upper half plane**

$$H = \{x + iy \in \mathbb{C} : \; y > 0\}$$

    is unbounded

Limit of a Sequence

- Let $\mathbb{Z}_+$ denote the set of positive integers
- A sequence is a function $S : \mathbb{Z}_+ \to \mathbb{C}$
- We usually denote $z_n = S(n)$ and write the sequence as

$$(z_n : \ n \in \mathbb{Z}_+)$$

- $z \in \mathbb{C}$ is the **limit** of a sequence $(z_n : \ n \in \mathbb{Z}_+)$ if for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{Z}_+$ such that

$$\forall n \geq N_\epsilon, \ z_n \in D(z_\infty, \epsilon)$$

- If so, we write

$$\lim_{n \to \infty} z_n = z$$

- The definition can also be stated as: $z \in \mathbb{C}$ is the limit of a sequence $(z_n : \ n \in \mathbb{Z}_+)$ if for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{Z}_+$ such that

$$\forall n \geq N_\epsilon, \ |z_n - z| \leq \epsilon$$

Convergent and Cauchy Sequences

- If a sequence has a limit, the limit is unique
- A sequence with a limit is called **convergent**
- A sequence $(z_n : \ n \in \mathbb{Z}_+)$ is **Cauchy** if for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{Z}_+$ such that

$$\forall m, n \geq N_\epsilon, \ |z_n - z_m| < \epsilon$$

- A sequence is Cauchy if and only if it is convergent