# MATH-GA2120 Linear Algebra II

## Linear Transformation of Ball is Ellipsoid
## Operator Norm of Linear Map
## Frobenius Norm of Linear Map
## Condition Number of Linear Map

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# Image of Unit Ball

- The closed unit ball centered at the origin in $\mathbb{R}^n$ is

$$B = \{x \in \mathbb{R}^n : \; x \cdot x \leq 1\}$$

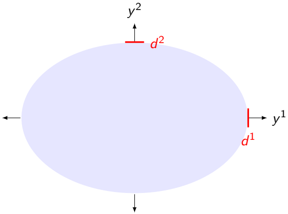
- Consider the image of $B$ under a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$
- If $A$ is diagonal, then if $y = Ax \in AB$,

$$Ay = A \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix} = \begin{bmatrix} d^1 & 0 & \cdots & 0 \\ 0 & d^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d^n \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix} = \begin{bmatrix} d^1 x^1 \\ d^2 x^2 \\ \vdots \\ d^n x^n \end{bmatrix}$$

- Therefore, $y \in AB$ if and only if

$$1 \geq (x^1)^2 + \cdots + (x^n)^2 = \left(\frac{y^1}{d^1}\right)^2 + \cdots + \left(\frac{y^n}{d^n}\right)^2$$

# Ellipse



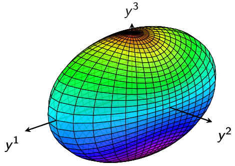

- If

$$y = \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} = \begin{bmatrix} d^1 & 0 \\ 0 & d^2 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = Ax$$

then

$$x \in B \iff \frac{(y^1)^2}{(d^1)^2} + \frac{(y^2)^2}{(d^2)^2} \leq 1$$

# 3-Dimensional Ellipsoid




$$\frac{(y^1)^2}{(d^1)^2} + \frac{(y^2)^2}{(d^2)^2} + \frac{(y^3)^2}{(d^3)^2} \le 1$$

# $n$-Dimensional Ellipsoid in $\mathbb{R}^n$

- Given $d^1, \dots, d^n \neq 0$,

$$E = \left\{(y^1, \dots, y^n) \in \mathbb{R}^n : \; \frac{(y^1)^2}{(d^1)^2} + \cdots + \frac{(y^n)^2}{(d^n)^2} \leq 1\right\}$$

  is called an $n$-dimensional **ellipsoid**

- If $A$ is a diagonal matrix with nonzero diagonal entries $d^1, \dots, d^n$, then

$$\begin{aligned} AB &= E \\ &= \{y \in \mathbb{R}^n : \; (A^{-1}y, A^{-1}y) \leq 1\} \end{aligned}$$

# Ellipsoids in Inner Product Space

- A subset $E$ of an $n$-dimensional real inner product space is an $n$-dimensional **ellipsoid** if there is a unitary basis $(u_1, \ldots, u_n)$ and nonzero scalars $d_1, \ldots, d_n$ such that

$$E = \left\{ y^1 u_1 + \cdots + y^n u_n : \ \frac{(y^1)^2}{(d^1)^2} + \cdots + \frac{(y^n)^2}{(d^n)^2} \leq 1 \right\}$$

- A subset $E$ of an $n$-dimensional realinner product space is an $k$-dimensional **ellipsoid** if there is a unitary set $(u_1, \ldots, u_k)$ and nonzero scalars $d_1, \ldots, d_k$ such that

$$E = \left\{ y^1 u_1 + \cdots + y^n u_k : \ \frac{(y^1)^2}{(d^1)^2} + \cdots + \frac{(y^k)^2}{(d^k)^2} \leq 1 \right\}$$

# Unitary Transformation of Ball is Ball

- If $X$ and $Y$ are inner product spaces with the same dimension, a map $U : X \to Y$ is a unitary transformation, if, for any $v \in X$,
$$(U(x), U(x))_Y = (x, x)_X$$
- Therefore, if
$$B_X = \{x \in X : \ (x, x) = 1\},$$
then
$$U(B_X) \subset B_Y$$
- On the other hand, if $y \in B_Y$, then $U^*(y)) \in B_X$ and $U(U^*(x)) = x$, which implies
$$B_Y \subset U(B_X)$$
- It follows that $U(B_X) = B_Y$

# Singular Value Decomposition

- Let $X$ and $Y$ be real inner product spaces such that $\dim(X) = m$ and $\dim(Y) = n$
- $L : X \to Y$ be a linear transformation
- The singular value decomposition of $L$ can be described as follows:
  - There exists a unitary basis $(e_1, \ldots, e_m)$ of $X$ and a unitary basis $(f_1, \ldots, f_n)$ of $Y$ such that if $r = \operatorname{rank}(L)$, then

    $$L(e_k) = \begin{cases} s_k f_k & \text{if } 1 \leq k \leq r \\ 0 & \text{if } r+1 \leq k \leq m \end{cases},$$

    where $s_1, \ldots, s_n$ are the singular values of $L$
  - In particular, $(e_1, \ldots, e_r)$ is a unitary basis of $(\ker(L))^\perp$ and $(f_1, \ldots, f_r)$ is a unitary basis of $\operatorname{image}(L)$

# Linear Transformation of Ball is an Ellipsoid (Part 1)

- The unit ball is

$$B = \{x^1 e_1 + \cdots + x^n e_n : \ (x^1)^2 + \cdots + (x^n)^2 \leq 1\}$$

- If $x \in B$, then

$$\begin{aligned} L(x) &= x^1 L(e_1) + \cdots + x^n L(e_n) \\ &= s_1 x^1 f_1 + \cdots + s_r x^r f_r \\ &= y^1 f_1 + \cdots + y^r f_r, \end{aligned}$$

where

$$\frac{(y^1)^2}{(s_1)^2} + \cdots + \frac{(y^r)^2}{(s_r)^2} = (x^1)^2 + \cdots + (x^r)^2 \leq 1$$

# Linear Transformation of Ball is an Ellipsoid (Part 2)

- The set

$$E = \left\{ y^1 f_1 + \cdots + y^r f_r : \frac{(y^1)^2}{(s_1)^2} + \cdots + \frac{(y^n)^2}{(s_r)^2} = (x^1)^2 + \cdots + (x^r)^2 \leq 1 \right\} \subset \text{image}(L)$$

is an $r$-dimensional ellipsoid in $Y$ such that

$$L(B_X) \subset E$$

# Linear Transformation of Ball is an Ellipsoid (Part 3)

- Conversely, if $y = y^1 f_1 + \cdots + y^r f_r \in E$, then
$$L(x) = y,$$
where
$$x = \left(\frac{y^1}{s_1}\right) e_1 + \cdots + \left(\frac{y^r}{s_r}\right) e_n \in B$$
- It follows that $E \subset L(B)$
- Therefore, $E = L(B)$

# Operator Norm of Linear Map

- Let $X$ and $Y$ be inner product spaces and $L : X \to Y$ be a linear map
- The **operator norm** of $L$ is defined to be

$$\|L\| = \sup\{|L(x)| \ : \ x \in B_X\}$$

- Let $s_1 \leq s_2 \leq \cdots \leq s_r$ be the singular values of $L$
- For any $x = x^1 e_1 + \cdots + x^m e_m \in B$,

$$\begin{aligned}(L(x), L(x)) &= (x^1 s_1 f_1 + \cdots + x^r s_r f_r, x^1 s_1 f_1 + \cdots + x^r s_r f_r) \\ &= (s_1)^2 (x^1)^2 + \cdots + (s_r)^2 (x^r)^2 \\ &\leq (s_r)^2 ((x^1)^2 + \cdots + (x^r)^2) \\ &\leq (s_r)^2\end{aligned}$$

- Moreover,

$$(L(e_r), L(e_r)) = (s_r f_r, s_r f_r) = (s_r)^2$$

- Therefore, $\|L\|$ is equal to the largest singular value of $L$

# Change of Basis Formula

- Let $L : X \to X$ be a linear endomorphism (codomain is domain)
- Given a basis $E(e_1, \ldots, e_m)$ of $X$, there is a matrix $M$ such that
$$L(e_k) = M^j_k e_j, i.e., L(E) = EM$$
- If $F = (f_1, \ldots, f_m)$ is another basis such that
$$f_k = A^j_k e_j,\ i.e., F = EA,$$
then
$$L(F) = L(EA) = L(E)A = EMA = FA^{-1}MA$$

# Trace of a Linear Endomorphism

- If $L(E) = EM$, then the trace of $L$ is defined to be

$$\operatorname{trace}(L) = M^1_1 + \cdots + M^m_m$$

- If $L(F) = EN$, then $N = A^{-1}MA$, i.e.,

$$N^l_k = (A^{-1})^l_i M^i_j A^j_k$$

- Therefore,

$$\begin{aligned}
N^1_1 + \cdots + N^m_m &= N^k_k \\
&= (A^{-1})^k_i M^i_j A^j_k \\
&= A^j_k (A^{-1})^k_i M^i_j \\
&= \delta^j_i M^i_j \\
&= M^j_j \\
&= M^1_1 + \cdots + M^m_m
\end{aligned}$$

- The definition of $\operatorname{trace}(L)$ does not depend on the basis used

# Frobenius Norm of a Linear Transformation

- Let $X$ and $Y$ be real inner product spaces
- Let $L : X \to Y$ be a linear map
- Recall that the adjoint of $L$ is the map $L^* : Y \to X$ such that for any $x \in X$ and $y \in Y$,

$$(L(x), y) = (x, L^*(y))$$

- The **Frobenius norm** or **Hilbert-Schmidt norm** of $L$ is defined to be $\|L\|_2$, where

$$\|L\|_2^2 = \mathrm{trace}(L^* L)$$

# Frobenius Norm With Respect to Basis

- Let $(e_1, \ldots, e_m)$ be a unitary basis of $X$ and $(f_1, \ldots, f_n)$ be a unitary basis of $Y$ such that

$$L(e_k) = \begin{cases} s_k f_k & \text{if } 1 \leq k \leq r \\ 0 & \text{if } r+1 \leq k \leq m, \end{cases}$$

- The adjoint of $L$ is given by

$$L^*(f_k) = \begin{cases} s_k e_k & \text{if } 1 \leq k \leq r \\ 0 & \text{if } r+1 \leq k \leq n \end{cases}$$

- Therefore,

$$L^*L(e_k) = \begin{cases} s_k^2 e_k & \text{if } 1 \leq k \leq r \\ 0 & \text{if } r+1 \leq k \leq m \end{cases}$$

- It follows that

$$\|L\|_2^2 = \text{trace}(L^*L) = s_1^2 + \cdots + s_r^2$$

- Observe that the operator norm is always less than or equal to the Frobenius norm,

# Solving a Linear System with Errors

- Let $L : X \to Y$ be a linear map between inner product spaces
- Suppose that, given $y \in Y$, we want to solve

$$L(x) = y,$$

  for $x$ but the exact value of $y$ is not known
- If the measured value of $y$ is $y + \Delta y$ and

$$x + \Delta x = L^{-1}(y + \Delta y),$$

  then

$$\Delta x = L^{-1}(\Delta y)$$

- The relative error of $x$ can ye estimated in terms of the relative error of $y$:

$$\frac{|\Delta x|}{|x|} = \frac{|L^{-1}(\Delta y)|}{|y|}\frac{|y|}{|x|} = \frac{|L^{-1}(\Delta y)|}{|y|}\frac{|L(x)|}{|x|} \leq \|L^{-1}\|\|L\|\frac{|\Delta y|}{|y|}$$

# Condition Number of Linear Map

- $\|L^{-1}\|\|L\|$ is the **condition number** of the linear map
- It shows how sensitive the error in $x$ is to the error in $y$
- A linear map is **ill-conditioned** if the condition number is large
- The condition number can be changed by changing the inner product

# Natural Isomorphism of Inner Product Space and Dual

- Let $V$ be an inner product space
- There is a natural map

$$\begin{aligned} \delta : V &\to V^* \\ w &\mapsto \ell_w, \end{aligned}$$

  where for any $v \in V$,

$$\langle \ell_w, v \rangle = (v, w)$$

- $w$ is in the kernel of this map if $\ell_w = 0$, i.e., for any $v \in V$,

$$0 = \langle \ell_w, v \rangle = (v, w)$$

  This holds if and only if $w = 0$