MATH-GA2120 Linear Algebra II

Adjoint Map
Fundamental Subspaces
Isometries
Unitary Maps
Unitarily Equivalent Maps

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Adjoint of a Linear Transformation and of a Matrix

- Let V, W be inner product spaces and $L : V \to W$ be a linear map
- The **(Hermitian) adjoint** of L is defined to be the map $L^* : W \to V$ such that for any $v \in V$ and $w \in W$,

$$(L(v), w) = (v, L^*(w))$$

- If M is an m-by-n matrix, its **(Hermitian) adjoint** is defined to be the n-by-m matrix

$$M^* = \overline{M}^T$$

- Let $(e_1, \dots, e_n)$ be a unitary basis of V and $(f_1, \dots, f_m)$ be a unitary basis of W
- If $L : V \to W$ is a linear map and M is the matrix such that for any $1 \le i \le n$,

$$L(e_i) = f_1 M_i^1 + \cdots + f_m M_i^m = f_a M_i^a m$$

then it is easy to verify that for any vectors

$$v = e_1 a^1 + \cdots + e_n a^n \text{ and } w = f_1 b^1 + \cdots + f_m b^m,$$

Basic Properties of Adjoint Map

- If $L, L_1, L_2 : V \to W$ are linear maps and $c \in \mathbb{F}$, then

$$(L_1 + L_2)^* = L_1^* + L_2^*$$
$$(cL)^* = \bar{c}L^*$$
$$(L_1 \circ L_2)^* = L_2^* \circ L_1^*$$
$$(L^*)^* = L$$
$$(w, L(v)) = (L^*(w), v)$$

Fundamental Subspaces of Adjoint Map

- Let $L : V \to W$ be a map between inner produt spaces
- Then

$$\ker(L^*) = (\text{image}(L))^\perp \tag{1}$$
$$\ker(L) = (\text{image}(L^*))^\perp \tag{2}$$
$$\text{image}(L) = (\ker(L^*))^\perp \tag{3}$$
$$\text{image}(L^*) = (\ker(L))^\perp \tag{4}$$

- That
  - For any subspace S, $(S^\perp)^\perp = S$
  - For any linear map A, $(A^*)^* = A$

  imply that (2),(3),(4) follow directly from (1)

Proof that $\ker(L^*) = (\text{image}(L))^\perp$

$$\begin{aligned} w \in \ker(L^*) &\iff L^*(w) = 0 \\ &\iff \forall v \in V,\ (v, L^*(w)) = 0 \\ &\iff \forall v \in V,\ (L(v), w) = 0 \\ &\iff w \in (\text{image}(L))^\perp \end{aligned}$$

Geometric Description of a Linear Map and its Adjoint

- Recall that if E is a subspace of V, then

$$V = E \oplus E^{\perp}$$

- Therefore,

$$V = (\ker(L)) \oplus (\ker(L))^{\perp}$$

- It is easy to show that the restriction of L to $(\ker(L))^{\perp}$,

$$L : (\ker(L))^{\perp} \to \operatorname{image}(L)$$

  is bijective
- Equivalently, by (4),

$$L : \operatorname{image}(L^*) \to \operatorname{image}(L)$$

  is bijective
- Therefore,

$$\operatorname{rank}(L) = \dim(\operatorname{image}(L)) = \dim(\operatorname{image}(L^*)) = \operatorname{rank}(L^*)$$

Isometries

- **(Corrected Version)** A map (not assumed to be linear) $L : V \to W$, where V and W are normed vector spaces, is an **isometry** if for any $v_1, v_2 \in V$,

$$|L(v_2 - v_1)| = |v_2 - v_1|$$

- **Theorem:** If V and W are inner product spaces and $L : V \to W$ is an isometry, then L is linear and satisfies for any $v_1, v_2 \in V$,

$$(L(v_1), L(v_2)) = (v_1, v_2)$$

- **Lemma:** $L : V \to W$ is an isometry if and only if $L^* \circ L = I_V$
- Therefore, L^* is a left inverse

Unitary Maps

- If $\dim(V) = \dim(W)$, then an isometry $L : V \to W$ is called a **unitary map**
- A map L is unitary if and only if $L^* \circ L = I_V$
- A unitary map L is invertible, and its inverse is L^*
- A matrix $M \in \mathrm{gl}(n, \mathbb{F})$ is **unitary** if

$$M^* M = I$$

- A unitary matrix M is invertible, and its inverse is M^*

Basic Properties of Isometries and Unitary Maps

- If $\dim(V) \leq \dim(W)$, $L : V \to W$ is an isometry, and $(v_1, \ldots, v_n)$ is an orthonormal basis, then $(L(v_1), \ldots, L(v_n))$ is an orthonormal set in W
- If $\dim(V) = \dim(W)$, $L : V \to W$ is unitary, and $(v_1, \ldots, v_n)$ is an orthonormal basis, then $(L(v_1), \ldots, L(v_n))$ is an orthonormal basis of W
- If L is unitary, then $L^{-1} = L^*$ is unitary
- If $L_1 : V \to W$ and $L_2 : W \to X$ are unitary, then so is $L_2 \circ L_1 : V \to X$

Examples of Unitary Matrices

- An n-by-n matrix is unitary if and only if its columns form a unitary basis of $\mathbb{F}^n$
- A real 2-by-2 matrix is a unitary matrix with positive determinant if and only if it is of the form

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- For any $\theta^1, \theta^2 \in \mathbb{R}$

$$\begin{bmatrix} e^{i\theta^1} & 0 \\ 0 & e^{i\theta^2} \end{bmatrix}$$

More Properties of Unitary Matrices

- Let U be a unitary matrix
- $\det(U^*) = \overline{\det(U)}$
  - Because $\det(A^T) = \det(A)$ and $\det(\overline{A}) = \overline{\det(A)}$
- If λ is an eigenvalue of U, then $|\lambda| = 1$
  - Because if λ is an eigenvalue of U with eigenvector v, then

$$|v| = |Uv| = |\lambda v| = |\lambda||v|,$$

    which implies $|\lambda| = 1$

Unitarily Equivalent Matrices

- $M_1, M_2 \in \mathrm{gl}(n, \mathbb{F})$ are **unitarily equivalent** if there exists a unitary matrix U such that
$$M_2 = UM_1U^*$$
- Since $U^* = U^{-1}$, unitarily equivalent implies similar
- **Fact:** A matrix M is unitarily equivialent to a diagonal matrix if and only if there is a unitary basis of eigenvectors
- If $A = UDU^*$, then the standard basis $(e_1, \ldots, e_n)$ are eigenvectors of D
- For each $1 \leq k \leq n$, let $f_k = U(e_k)$
- For each $1 \leq j, k \leq n$,
$$(f_j, f_k) = (U(e_j), U(e_k)) = (e_j, e_k) = \delta_{jk}$$
- Moreover, if $De_k = \lambda_k e_k$, then
$$Mf_k = UDU^*Ue_k = UDe_k = U(\lambda_k e_k) = \lambda_k Ue_k = \lambda_k f_k$$
- Therefore, $(f_1, \ldots, f_n)$ is a unitary basis of eigenvectors
- Converse is even easier