MATH-GA2120 Linear Algebra II

Triangle Inequality
Polarization Inequality
Norms
Orthogonal Set and Basis
Orthogonal Decomposition and Projection
Gram-Schmidt Construction of Orthonormal Basis

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Proof of Triangle Inequality

- The triangle inequality follows easily from Cauchy-Schwarz inequality

$$
\begin{aligned}
|v+w|^2 &= (v+w, v+w) \\
&= |v|^2 + (v,w) + (w,v) + |w|^2 \\
&\leq |v|^2 + |(v,w)| + |(w,v)| + |w|^2 \\
&\leq |v|^2 + 2|v||w| + |w|^2 \\
&= (|v| + |w|)^2
\end{aligned}
$$

- If $|v+w| = |v| + |w|$, then

$$|(v,w)| = |(v,w)| = |v||w|,$$

which implies $v = tw$ and therefore

$$|t+1|^2|w|^2 = |tw+w|^2 \quad = |tw|^2 + |w|^2 = (|t|^2+1)|w|^2,$$

which implies that $t = \bar{t}$, i.e., $t \in \mathbb{R}$

Polarization Identities

- On $\mathbb{R}^n$

$$(v, w) = \frac{1}{4}(|v + w|^2 - |v - w|^2)$$

- On $\mathbb{C}^n$

$$(v, w) = \frac{1}{4}(|v + w|^2 + i|v + iw|^2 - |v - w|^2 - i|v - iw|^2)$$

Norm Defined by Inner Product

- The norm of $v \in V$,

$$|v| = \sqrt{(v, v)}$$

satisfies the following properties for any $s \in \mathbb{F}$, $v, w \in V$

$$\begin{aligned} |sv| &= |s||v| && \text{(Homogeneity)} \\ |v| &\geq 0 && \text{(Nonnegativity)} \\ |v| &= 0 \iff v = 0 && \text{(Nondegeneracy)} \\ |v + w| &\leq |v| + |w| && \text{(Triangle inequality)} \end{aligned}$$

- Homogeneity and the triangle inequality imply convexity: For any $0 \leq t \leq 1$ and $v, w \in V$,

$$|(1-t)v + tw| \leq (1-t)|v| + t|w|$$

Norm

- A norm on a vector space V over $\mathbb{F}$ is a function

$$g : V \to \mathbb{R},$$

that satisfies for any $s \in \mathbb{F}$ and $v, w \in V$,

$$\begin{aligned} |sv| &= |s||v| && \text{(Homogeneity)} \\ |v| &\geq 0 && \text{(Nonnegativity)} \\ |v| &= 0 \iff v = 0 && \text{(Nondegeneracy)} \\ |v + w| &\leq |v| + |w| && \text{(Triangle inequality)} \end{aligned}$$

Examples of Norms

- Given $1 \leq p < \infty$, the ℓ_p norm of $v \in \mathbb{F}^n$ is defined to be

$$|v|_p = (|v^1|^p + \cdots + |v^n|^p)^{1/p}$$

- The ℓ_∞ norm of $v \in \mathbb{F}^n$ is defined to be

$$|v|_\infty = \max(|v^1|, \ldots, |v^n|) = \lim_{p \to \infty} |v|_p$$

- The L_p norm of a continuous function $f : [0,1] \to \mathbb{C}$ is defined to be

$$\|f\|_p = \left(\int_{x=0}^{x=1} |f(x)|^p \, dx \right)^{1/p}$$

- The L_∞ norm of a continuous function $f : [0,1] \to \mathbb{C}$ is defined to be

$$\|f\|_\infty = \sup\{|f(x)| \; : \; 0 \leq x \leq 1\} = \lim_{p \to \infty} \|f\|_p$$

Parallelogram Identity

- A norm $|\cdot|$ on a vector space V satisfies the parallelogram identity

$$|v+w|^2 + |v-w|^2 = 2(|v|^2 + |w|^2), \ \forall v, w \in V$$

if and only if there is an inner product on V such that

$$|v|^2 = (v, v)$$

Orthogonality For Standard Dot Product on $\mathbb{R}^n$

- The following are synonyms: orthogonal, perpendicular, normal
- On $\mathbb{R}^n$,
  - Two vectors v_1, v_2 are called **orthogonal** if

$$v_1 \cdot v_2 = 0$$

  - A basis $(v_1, \ldots, v_n)$ is called **orthonormal** if for any $1 \leq i, j \leq n$,

$$v_i \cdot v_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Orthogonality on an Inner Product Space

- Let V be an n-dimensional vector space over $\mathbb{F}$ with inner product $(\cdot, \cdot)$
- Two vectors v_1, v_2 are **orthogonal** if

$$(v_1, v_2) = 0$$

- Vectors $v_1, \ldots, v_k$ are **mutually orthogonal** if for every $1 \leq i < j \leq k$,

$$(v_i, v_j) \neq 0$$

- Mutually orthogonal vectors must all be nonzero
- A set of muturally orthogonal vectors is called an **orthogonal set**

Linear Independence of Orthogonal Set

- An orthogonal set is linearly independent, because if
$$a^1 v_1 + \cdots + a^k v_k = 0,$$
then for any $1 \le j \le k$,
$$0 = (v_j, a^1 v_1 + \cdots + a^k v_k) = a^j (v_j, v_j)$$
Since $v_j \neq 0$, $(v_j, v_j) \neq 0$ and therefore $a^j = 0$
- If
$$v = a^1 v_1 + \cdots + a^k v_k,$$
then for each $1 \le j \le k$,
$$a^j = \frac{(v, v_j)}{|v_j|}$$
and
$$v = \frac{(v, v_1)}{|v_1|} + \cdots + \frac{(v, v_k)}{|v_k|}$$
- Any orthogonal set of n vectors is a basis

Orthonormal Set and Basis

- $\{v_1, \ldots, v_k\} \subset V$ is called an **orthonormal** set if for any $1 \le i, j \le k$,
$$(v_i, v_j) = \delta_{ij}$$
- If $\mathbb{F} = \mathbb{C}$, such a set is also called a **unitary** set
- An orthonormal set of n elements is called an **orthonormal** or **unitary** basis
- Any orthogonal set $\{v_1, \ldots, v_k\}$ can be turned into an orthonormal set,
$$\{\frac{v_1}{|v_1|}, \ldots, \frac{v_k}{|v_k|}$$
- An orthormal or unitary basis is an orthonormal set with n elements,
$$E = (e_1, \ldots, e_n) \subset V$$
- If $v = a^1 e_1 + \cdots + a^n e_n$, then
$$a_j = (v, e_j)$$
- I.e.,
$$v = (v, e_1)e_1 + \cdots + (v, e_n)e_n$$

Example: Finite Fourier Decomposition (Part 1)

- For each $-N \leq k \leq N$, consider

$$\begin{aligned} v_k : [0, 2\pi] &\to \mathbb{C} \\ \theta &\mapsto e^{ik\theta} \end{aligned}$$

- Let

$$V = \{a^{-N} v_N + \cdots + a^0 + \cdots + a^N v_N \ : \ (a^1, \ldots, a^N) \in \mathbb{C}^{2N+1}\}.$$

- V is a $(2N+1)$-dimensional complex vector space
- Consider the inner product

$$(f_1, f_2) = \int_{\theta=0}^{\theta=2\pi} f_1(\theta) \bar{f}_2(\theta) \, d\theta$$

Finite Fourier Decomposition (Part 2)

- If $j \neq k$, then

$$\begin{aligned}(v_j, v_k) &= \int_{\theta=0}^{\theta=2\pi} e^{i(j-k)\theta}\, d\theta \\ &= \left.\frac{e^{i(j-k)\theta}}{i(j-k)}\right|_{\theta=0}^{\theta=2\pi} \\ &= 0 \\ (v_k, v_k) &= \int_{\theta=0}^{\theta=2\pi} 1\, d\theta \\ &= 2\pi\end{aligned}$$

- Therefore, $(v_{-N}, \ldots, v_N)$ is an orthogonal basis, and $(u_{-N}, \ldots, u_N)$, where

$$u_k = \frac{v_k}{\sqrt{2\pi}},\ -N \leq k \leq N,$$

is an orthonormal basis

Finite Fourier Decomposition (Part 3)

- Given any $f : C^0([0, 2\pi])$, let

$$f_N(\theta) = a^{-N} u_{-N} + \cdots + a^N u_N,$$

where

$$a^k = (f, u_k) = \frac{1}{\sqrt{2\pi}} \int_{\theta=0}^{\theta_{2\pi}} f(\theta) e^{-ik\theta} \, d\theta$$

- When is f_N is a good approximation to f?
- When is

$$f = \sum_{k=-\infty}^{k=\infty} a^k u_k?$$

Orthogonal Complement

- Let V be a vector space with inner product $(\cdot, \cdot)$
- Given a subspace $E \subset V$, define its **orthogonal complement** to be the subspace

$$E^{\perp} = \{v \in V \ : \ \forall e \in E, (v, e) = 0\}$$

- $E \cap E^{\perp} = \{0\}$, because if

$$v \in E \cap E^{\perp},$$

then

$$|v|^2 = (v, v) = 0,$$

- If $v_1, v_2 \in E$, $w_1, w_2 \in E^{\perp}$, and

$$v_1 + w_1 = v_2 + v_2,$$

then

$$v_1 - v_2 = w_2 - w_1 \in E \cap E^{\perp}$$

and therefore, $v_1 = v_2$ and $w_1 = w_2$
- If follows that $E \oplus E^{\perp}$ is a subspace of V

Orthogonal Decomposition

- For each $v \in E \oplus E$, there exist unique $v_1 \in E$ and $v_2 \in E^\perp$ such that
$$v = v_1 + v_2$$
- Define the orthogonal projection maps
$$\begin{aligned} P_E : E \oplus E^\perp &\to E \\ v &\mapsto v_1 \end{aligned}$$
and
$$\begin{aligned} P_E^\perp : E \oplus E^\perp &\to E^\perp \\ v &\mapsto v_2 \end{aligned}$$

Orthogonal Projection Maps

- $P_E, P_E^\perp$ are linear maps
- $P_E : E \oplus E^\perp \to E$ is projection onto E:

$$\forall v \in E,\ P_E(v) = v$$

- $P_E^\perp : E \oplus E^\perp \to E^\perp$ is projection onto $E^\perp$:

$$\forall v \in E^\perp,\ P_E^\perp(v) = v$$

- Orthogonal decomposition: For any $v \in E \oplus E^\perp$,

$$P_E(v) \in E$$
$$P_E^\perp(v) \in E^\perp$$
$$v = P_E(v) + P_E^\perp(v)$$

Orthogonal Projection Minimizes Distance to a Subspace

- Observe that $v - P_E(v) = P_E^{\perp}(v) \in E^{\perp}$
- **Fact:** For each $v \in E \oplus E^{\perp}$ and $w \in E$,

$$|v - P_E(v)| \leq |v - w|$$

  and equality holds if and only if $w = P_E(v)$
- **Proof:** Let $v = v_1 + v_2$, where

$$v_1 = P_E(v) \in E \text{ and } v_2 = v - P_E(v) \in E^{\perp}$$

- Then for any $w \in E$,

$$\begin{aligned} |v - w|^2 &= |v - P_E(v) + P_E(v) - w|^2 \\ &= (v_2 + (v_1 - w), v_2 + (v_1 - w)) \\ &= (v_2, v_2) + 2(v_1 - w, v_2) + (v_1 - w, v_1 - w) \\ &\geq |v - P_E(v)|^2 \end{aligned}$$

  and equality holds if and only if

$$|v_1 - w, v_1 - w|^2 = (v_1 - w, v_1 - w) = 0$$

Orthogonal Projection Using an Orthonormal Set (Part 1)

- Let $(u_1, \ldots, u_k)$ be an orthonormal basis of a subspace $E \subset V$
- For any $v \in E$, there exist $a^1, \ldots, a^k \in \mathbb{F}$ such that

$$v = a^1 u_1 + \cdots + a^k u_k$$

- Since, for each $1 \leq j \leq k$,

$$(v, u_j) = (a^1 u_1 + \cdots + a^k u_k, u_j) = a^j,$$

  it follows that

$$v = (v, u_1)u_1 + \cdots + (v, u_k)u_k$$

Orthogonal Projection Using an Orthonormal Set (Part 2)

- Consider the map $\pi_E : V \to E$ given by

$$\pi_E(v) = (v, u_1)u_1 + \cdots + (v, u_k)u_k$$

- For any $v \in V$ and $1 \leq j \leq k$,

$$(v - \pi_E(v), u_k) = (v, u_k) - (v, u_k) = 0$$

  and therefore

$$v - \pi_E^\perp(v) \in E^\perp$$

- Therefore, if for any v,

$$\pi_E^\perp(v) = v - \pi_E(v),$$

  then

$$v = \pi_E(v) + \pi_E^\perp(v)$$

- It follows that, if E has an orthonormal basis, then

$$E \oplus E^\perp = V$$

Constructing an Orthonormal Basis of V (Part 1)

- Let E be a k-dimensional subspace of V, with $k \geq 1$
- Let $(v_1, \ldots, v_k)$ be a basis of E
- For each $1 \leq j \leq k$, let

$$E_j = \text{span}(v_1, \ldots, v_j)$$

- We can construct an orthonormal set that spans E by induction
- Let

$$u_1 = \frac{v_1}{|v_1|},$$

- Then $\{u_1\}$ is an orthonormal basis of E_1

Constructing an Orthonormal Basis (Part 2)

- Assume that $j < k$ and that $(u_1, \ldots, u_j)$ is an orthonormal basis of $E_j \subset E$
- Let
$$v_{j+1} = \pi_{E_j}(v_{j+1}) + \pi^{\perp}_{E_j}(v_{j+1}),$$
where
$$\begin{aligned} \pi_{E_j}(v_{j+1}) &= (v_{j+1}, u_1)u_1 + \cdots + (v_{j+1}, u_j)u_j \in E_j \\ \pi^{\perp}_{E_j}(v_{j+1}) &= v_{j+1} - \pi_{E_j}(v_{j+1}) \in E_j^{\perp} \end{aligned}$$
- Since $v_{j+1} \notin E_j$ and $\pi_{E_j}(v_{j+1}) \in E$, it follows that
$$\pi^{\perp}_{E_j}(v_{j+1}) \neq 0$$
- Let
$$u_{j+1} = \frac{\pi^{\perp}_{E}(v_{j+1})}{|\pi^{\perp}_{E}(v_{j+1})|}$$
- Since $u_{j+1} \in E_j^{\perp}$, $(u_{j+1}, u_i) = 0$ for all $1 \leq i \leq j$
- Therefore, $(u_1, \ldots, u_{j+1})$ is an orthonormal basis of E_{j+1}

Gram-Schmidt Construction of Orthonormal Basis

- Let $(v_1, \ldots, v_n)$ be a basis of an inner product space V
- There exists an orthonormal basis $(u_1, \ldots, u_n)$ such that for each $1 \leq k \leq n$,

$$\text{span}(u_1, \ldots, u_k) = \text{span}(v_1, \ldots, v_k)$$