MATH-GA1002 Multivariable Analysis

Geometry of Surface in Euclidean 3-Space

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3-Dimensional Euclidean Vector Space

- Let $\mathbb{V}$ be $\mathbb{R}^3$ viewed as a vector space with with the standard orientation, where the following are valid operations:
  - **Scaling**: Given $s \in \mathbb{R}$ and $v = (v^1, v^2, v^3) \in \mathbb{V}$, the vector obtained by scaling v by a factor s is

$$sv = (sv^1, sv^2, sv^3)$$

  - **Vector addition:** The sum of the vectors

$$v_1 = (v_1^1, v_1^2, v_1^3) \text{ and } v_2 = (v_2^1, v_2^2, v_2^3),$$

    is

$$v_1 + v_2 = (v_1^1 + v_2^1, v_1^2 + v_2^2, v_1^3 + v_2^3)$$

  - The **dot product** of $v_1, v_2 \in \mathbb{V}$

$$v_1 \cdot v_2 = v_1^1 v_2^1 + v_1^2 v_2^2 + v_1^3 v_2^3$$

  - The **length** of a vector v is

$$|v| = \sqrt{v \cdot v}$$

Euclidean 3-Space

- Let $\mathbb{E}$ denote $\mathbb{R}^3$ viewed as a set of points where the following are valid operations:
  - **Difference of two points**: Given points $x_0, x_1 \in \mathbb{E}$, there is vector $v \in \mathbb{V}$ that starts at p_0 and ends at p_1, where

    $$v = x_1 - x_0 = (x_1^1 - x_0^1, x_1^2 - x_0^2, x_1^3 - x_0^3)$$

  - **Point-vector addition:** Given a point x_0 and a vector $v \in \mathbb{V}$, there is a point x_1 such that $x_1 - x_0 = v$,

    $$x_1 = x_0 + v = (x_0^1 + v^1, x_0^2 + v^2, x_0^3 + v^3)$$

  - The **distance between two points** $x_0, x_1 \in \mathbb{E}$ is

    $$d(x_0, x_1) = |x_1 - x_0|$$

- For each $x \in \mathbb{E}$, there is a natural isomorphism

  $$T_x\mathbb{E} = \mathbb{V},$$

  where $T_x\mathbb{E}$ is the space of all possible velocity vectors of curves passing through x

Surface in $\mathbb{E}$

- $S \subset \mathbb{E}$ is a **parameterized surface** if there exists an open $U \subset \mathbb{R}^2$ and a smooth embedding $\Phi : U \to \mathbb{E}$ such that $S = \Phi(U) \subset \mathbb{E}$
- $S \subset \mathbb{E}$ is a **surface** if for each $p \in S$, there exists an open $O \subset \mathbb{E}$ such that $S \cap O$ is a parameterized surface
- A parameterization of $S \cap O$ is called a **local parameterization**

Surface as Level Set

- If $O \subset \mathbb{E}$ is open and $f : O \to \mathbb{R}$ is smooth, then for each $h \in \mathbb{R}$,
$$f^{-1}(h) = \{x \in O \ : \ f(x) = h\}$$
is called a **level set**
- If for each $x \in f^{-1}(h)$, $df(x) \neq 0$, then $f^{-1}(h)$ is a surface
- S is a surface if and only if for each $p \in S$, there is an open $O \subset \mathbb{E}$ such that $S \cap O$ is a level set

Examples

- If $D \subset \mathbb{R}^2$ is open, the graph of $f : D \to \mathbb{R}$,
$$S = \{(x, y, f(x, y)) \ : \ (x, y) \in D\}$$
is a surface
- The set
$$S = \{(x, y, z) \in \mathbb{E} \ : \ x^2 + y^2 + z^2 = 1\}$$
is a surface
- The boundary of a 3-dimensional rectangle
$$R = [a^1, b^1] \times [a^2, b^2] \times [a^3, b^3]$$
is not a surface
- The following subset of the boundary of R is a surface
$$\begin{aligned} S = &(\{a^1, b^1\} \times (a^2, b^2) \times (a^3, b^3)) \\ &\cup ((a^1, b^1) \times \{a^2, b^2\} \times (a^3, b^3)) \\ &\cup ((a^1, b^1) \times (a^2, b^2) \times \{a^3, b^3\}) \end{aligned}$$
is a surface

Tangent Space of Surface

- For each $x_0 \in S$, let $x : U \to S \subset \mathbb{E}$ be a parameterization of S in a neighborhood of x_0 such that $x(0) = x_0$
- The pushforward of x at each $u \in U$ is a linear map

$$x_u : T_u U \to T_{x(u)} \mathbb{E}$$

- Since the map $x : U \to S$ is an embedding, the pushforward is injective
- Recall that $x_u(T_u U)$ is the space of all possible velocity vectors of curves passing through $x(u)$
- The **tangent space** of S at $x(u)$ is

$$T_{x(u)} S = x_u(T_u U) \subset T_{x(u)} \mathbb{E}$$

Tangent and Cotangent Bundle

- The **tangent bundle of a surface** S is

$$T_*S = \coprod_{x \in S} T_x S$$

- A **vector field** is a map

$$v : S \to T_*S$$

such that for each $x \in S$, $v(x) \in T_x S$

- The **cotangent bundle of a surface** S is

$$T^*S = \coprod_{x \in S} T_x^* S$$

- A **differential** 1-**form** is a map

$$\theta : S \to T^*S$$

such that for each $x \in S$, $\theta(x) \in T_x^* S$

Differential 2-Form on a Surface

- The **exterior** 2-**tensor bundle** of S is

$$\Lambda^2 T^* S = \coprod_{x \in S} \Lambda^2 T_x^* S$$

- A differential 2-form on a surface S is a map

$$\Theta : S \to \Lambda^2 T^* S,$$

such that

$$\Theta(x) \in \Lambda^2 T_x^* S$$

Pullback of Differential Forms

- Let S and S' be surfaces and $F : S \to S'$ be a smooth map
- Recall that given a linear map

$$F_x : T_x S \to T_{F(x)} S',$$

its dual map is the pullback

$$F^x : T^*_{F(x)} S' \to T^*_x S$$

- The pullback of a differential form Θ on S' is the differential form $F^*\Theta$ on S, where for each $x \in S$,

$$(F^*\Theta)(x) = F^x(\Theta(F(x)))$$

- If θ is a 1-form, then for each $v \in T_x S$,

$$\langle v, (F^*\theta)(x)\rangle = \langle F_x v, \theta(F(x))\rangle$$

Orientation of a Surface

- Any basis (e_1, e_2) of T_xS defines an orientation of T_xS
- A parameterization $x : U \subset \mathbb{E}$ of S defines an orientation on T_xS, for each $x \in S$, by using the basis $(\partial_1 x(u), \partial_2 x(u))$, where $x = x(u)$
- If $\nu \in T_x\mathbb{E}$ is **not** tangent to S at x, then it uniquely determines an orientation
- A basis (e_1, e_2) of T_xS is positively oriented if (ν, e_1, e_2) is a positively oriented basis of $\mathbb{E}$, using the standard orientation

Rectangular Surface

- Let $R \subset \mathbb{R}^2$ be a rectangle and $\mathring{R} = R \backslash \partial R$ be its interior
- A smooth map
$$x : R \to \mathbb{E}$$
is a **rectangular parameterization** of S if $x(R) = S$ and the map
$$x|_{\mathring{R}} : \mathring{R} \to \mathbb{E}$$
is an embedding
- A surface is **rectangular** if it has a rectangular parameterization

Orthonormal Frame

- An **orthonormal frame** is a basis (e_1, e_2, e_3) of $\mathbb{V}$ such that

$$e_j \cdot e_k = \delta_{jk}$$

- An orthonormal frame can be written as a row matrix of vectors or a matrix whose columns are the three vectors in the frame,

$$\begin{aligned} E &= (e_1, e_2, e_3) \\ &= \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \\ &= \begin{bmatrix} e_1^1 & e_2^1 & e_3^1 \\ e_1^2 & e_2^2 & e_3^2 \\ e_1^3 & e_2^3 & e_3^3 \end{bmatrix} \\ &= \begin{bmatrix} \partial_1 & \partial_2 & \partial_3 \end{bmatrix} \begin{bmatrix} e_1^1 & e_2^1 & e_3^1 \\ e_1^2 & e_2^2 & e_3^2 \\ e_1^3 & e_2^3 & e_3^3 \end{bmatrix} \end{aligned}$$

Orthonormal Coframe

- The dual coframe is the dual basis of $\mathbb{V}^*$,

$$\begin{aligned} E^* &= (\omega^1, \omega^2, \omega^3) \\ &= \begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{bmatrix} \\ &= \begin{bmatrix} \omega^1_1\, dx^1 + \omega^1_2\, dx^2 + \omega^1_3\, dx^3 \\ \omega^2_1\, dx^1 + \omega^2_2\, dx^2 + \omega^2_3\, dx^3 \\ \omega^3_1\, dx^1 + \omega^3_2\, dx^2 + \omega^3_3\, dx^3 \end{bmatrix} \\ &= \begin{bmatrix} \omega^1_1 & \omega^1_2 & \omega^1_3 \\ \omega^2_1 & \omega^2_2 & \omega^2_3 \\ \omega^3_1 & \omega^3_2 & \omega^3_3 \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \end{bmatrix}, \end{aligned}$$

  where

$$\langle \omega^j, e_k \rangle = \delta^j_k$$

- Using matrix notation,

$$\langle E^*, E \rangle = I$$

Parameterized Surface in Coordinates

- Let $U \subset \mathbb{R}^2$ be open and

$$x : U \to \mathbb{E}$$

  be a parameterized surface
- Denote $u = (u^1, u^2) \in U$ and $x = (x^1, x^2, x^3)$, where each

$$x^k : U \to \mathbb{R}$$

  is a scalar function
- By the definition of a parameterized surface, if $u \in U$, $v = (v^1, v^2) \in T_u U$, then the pushforwrd map

$$x_u : T_u U \to T_{x(u)}\mathbb{E}$$

  is injective

Coordinate Vector Fields and 1-Forms

- The coordinate vector fields are the columns of the matrix

$$\begin{bmatrix} \partial_1 x & \partial_2 x \end{bmatrix} = \begin{bmatrix} \partial_1 x^1 & \partial_2 x^2 \\ \partial_1 x^2 & \partial_2 x^2 \\ \partial_1 x^3 & \partial_2 x^3 \end{bmatrix}$$

  are linearly independent

- The coordinate 1-forms are

$$dx = \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \end{bmatrix} = \begin{bmatrix} du^1 \partial_1 x^1 + du^2 \partial_2 x^1 \\ du^1 \partial_1 x^2 + du^2 \partial_2 x^2 \\ du^1 \partial_1 x^1 + du^3 \partial_2 x^3 \end{bmatrix} = \begin{bmatrix} \partial_1 x^1 & \partial_2 x^2 \\ \partial_1 x^2 & \partial_2 x^2 \\ \partial_1 x^3 & \partial_2 x^3 \end{bmatrix} \begin{bmatrix} du^1 \\ du^2 \end{bmatrix}$$

Orthonormal Moving Frame on Surface

- An **orthonormal moving frame** on a parameterized surface $x : U \to S$ consists of 3 vector-valued maps

$$e_k : U \to \mathbb{V}, \; k = 1, 2, 3,$$

such that for each $u \in U$,

$$e_j(u) \cdot e_k(u) = \delta_{jk}$$

- We can write the moving frame as a row matrix of vector fields or a matrix whose columns are the vector fields,

$$E = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} e_1^1 & e_2^1 & e_3^1 \\ e_1^2 & e_2^2 & e_3^2 \\ e_1^3 & e_2^3 & e_3^3 \end{bmatrix}$$

Orthonormal Moving Dual Coframe

- The **orthonormal moving dual coframe** consists of a column matrix of 1-forms,

$$E^* = \begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{bmatrix}$$

such that for each $u \in U$,

$$\langle \omega^j(u), e_k(u) \rangle = \delta^j_k,$$

i.e.,

$$\langle E^*, E \rangle = I$$

Orthonormal Coframe Using Dot Product

- Consider the 1-forms

$$\theta^k = e_k \cdot dx = e_k^1\, dx^1 + e_k^2\, dx^2 + e_k^3\, dx^3$$

- They satisfy

$$\begin{aligned}\langle e_j, \theta^k \rangle &= \langle e_j^1 \partial_1 + e_j^2 \partial_2 + e_j^3 \partial_3, e_k^1\, dx^1 + e_k^2\, dx^2 + e_k^3\, dx^3 \rangle \\ &= e_j^1 e_k^1 + e_j^2 e_k^2 + e_j^3 e_k^3 \\ &= e_j \cdot e_k \\ &= \delta_{jk}\end{aligned}$$

- Therefore, $(\theta^1, \theta^2, \theta^3)$ is the dual coframe of (e_1, e_2, e_3)
- It follows that

$$(\omega^1, \omega^2, \omega^3) = (e_1 \cdot dx, e_2 \cdot dx, e_3 \cdot dx)$$

$dx : \mathbb{V} \to \mathbb{V}$ Is the Identity Map

- For each $u \in U$, the differential of the function $x^k : U \to \mathbb{E}$ is the pullback of the differential of the coordinate function $x^k : \mathbb{E} \to \mathbb{R}$
- On $\mathbb{E}$, the map

$$x = (x^1, x^2, x^3) : \mathbb{E} \to \mathbb{E}$$

  is the identity map
- If $v \in \mathbb{V}$, then

$$\langle v, dx \rangle = \begin{bmatrix} \langle v, dx^1 \rangle \\ \langle v, dx^2 \rangle \\ \langle v, dx^3 \rangle \end{bmatrix} = \begin{bmatrix} \langle v^1\partial_1 + v^2\partial_2 + v^3\partial_3, dx^1 \rangle \\ \langle v^1\partial_1 + v^2\partial_2 + v^3\partial_3, dx^2 \rangle \\ \langle v^1\partial_1 + v^2\partial_2 + v^3\partial_3, dx^3 \rangle \end{bmatrix} = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = v$$

- In other words, the differential of the identity map $x : \mathbb{E} \to \mathbb{R}$ is the identity map

$$dx : \mathbb{V} \to \mathbb{V}$$

dx With Respect to Orthonormal Frame

- On the other hand, at each $x(u) \in S$, the frame $E(u) = (e_1(u), e_2(u), e_3(u))$ and its dual frame $E^*(u) = (\omega^1(u), \omega^2(u), \omega^3(u))$ satisfies

$$EE^* = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{bmatrix} = e_k \omega^k$$

defines a map $\mathbb{V} \to \mathbb{V}$, where if $v = e_j v^j$,

$$\begin{aligned} \langle v, EE^* \rangle &= \langle e_j v^j, e_k \omega^k \rangle \\ &= v^j e_k \langle e_j, \omega^k \\ &= v^j e_k \delta_j^k \\ &= v^k e_k \\ &= v \end{aligned}$$

- In other words, for each $u \in U$, the map $EE^* : \mathbb{V} \to$ is just the identity map and therefore

$$dx = EE^* = e_k \omega^k$$

Connection 1-Forms on $\mathbb{E}$

- Consider the 1-forms

$$\omega^j_k = e_j \cdot de_k = e^1_j \, de^1_k + e^2_j \, e^2_k + e^3_j \, de^3_k$$

- Then since $EE^T = EE^{-1} = E^{-1}E = E^TE = I$,

$$\begin{aligned} e_j\omega^j_k &= e_j(e^1_j \, de^1_k + e^2_j \, e^2_k + e^3_j \, de^3_k) \\ &= \begin{bmatrix} e^1_1 & e^1_2 & e^1_3 \\ e^2_1 & e^2_2 & e^2_3 \\ e^3_1 & e^3_2 & e^3_3 \end{bmatrix} \begin{bmatrix} e^1_1 & e^2_1 & e^3_1 \\ e^1_2 & e^2_2 & e^3_2 \\ e^1_3 & e^2_3 & e^3_3 \end{bmatrix} \begin{bmatrix} de^1_k \\ de^2_k \\ de^3_k \end{bmatrix} \\ &= EE^T \, de_k \\ &= de_k \end{aligned}$$

- Therefore,

$$de_k = e_j\omega^j_k, \text{ i.e., } dE = E\Gamma,$$

where

$$\Gamma = \begin{bmatrix} \omega^1_1 & \omega^1_2 & \omega^1_3 \\ \omega^2_1 & \omega^2_2 & \omega^2_3 \\ \omega^3_1 & \omega^3_2 & \omega^3_3 \end{bmatrix}$$

Matrix of Connection 1-Forms is Skew-Symmetric

- Since

$$0 = e_j \cdot e_k = e_j^1 e_k^1 + e_j^2 e_k^2 + e_j^3 e_k^3,$$

  it follows that

$$\begin{aligned} 0 &= d(e_j \cdot e_k) \\ &= d(e_j^1 e_k^1 + e_j^2 e_k^2 + e_j^3 e_k^3) \\ &= de_j^1 e_k^1 + e_j^1 de_k^1 + de_j^2 e_k^2 + e_j^2 de_k^2 + de_j^3 e_k^3 + e_j^3 de_k^3 \\ &= de_j \cdot e_k + e_j \cdot de_k \\ &= \omega_k^j + \omega_j^k \end{aligned}$$

- Therefore,

$$\Gamma^T = -\Gamma^T$$

Geometric Interpretation of Connection 1-Forms

- If $v \in T_x S$, then

$$\begin{aligned}\langle v, \omega_1^2 \rangle &= \langle v, e_2 \cdot de_1 \rangle \\ &= e_2 \cdot \langle v, de_1 \rangle\end{aligned}$$

  measures as x moves in the direction v, how quickly e_1 is turning towards e_2 in $T_x S$

- If $v \in T_x S$, then

$$\begin{aligned}\langle v, \omega_3^1 \rangle &= \langle v, e_1 \cdot de_3 \rangle \\ &= e_1 \cdot \langle v, de_3 \rangle\end{aligned}$$

  measures as x moves in the direction v, how quickly e_3 is turning towards e_1

First Structure Equation

- On one hand,

$$d(dx) = \begin{bmatrix} d(dx^1) \\ d(dx^2) \\ d(dx^3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

- On the other hand,

$$\begin{aligned} dx &= d(e_k \omega^k) \\ &= de_k \wedge \omega^k + e_k \, d\omega^k \\ &= e_j \omega^j_k \wedge \omega^k + e_j d\omega^j \\ &= e_j(\omega^j_k \wedge \omega^k + d\omega^j) \end{aligned}$$

- Therefore, for each $1 \leq j \leq 3$,

$$d\omega^j + \omega^j_k \wedge \omega^k = 0$$

Second Structure Equation

- Since

$$de_k = e_j \omega^j_k,$$

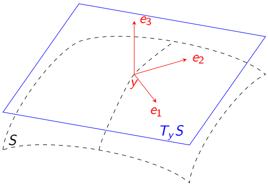
it follows that

$$\begin{aligned} 0 &= d(de_k) \\ &= d(e_j \omega^j_k) \\ &= de_j \wedge \omega^j_k + e_j d\omega^j_k \\ &= e_i \omega^i_j \wedge \omega^j_k + e_j d\omega^j_k \\ &= e_j (\omega^j_i \wedge \omega^i_k + d\omega^j_k) \end{aligned}$$

- Therefore,

$$d\omega^j_k + \omega^j_i \wedge \omega^i_j = 0$$

Adapted Orthonormal Moving Frame along a Surface



- An **adapted orthonormal moving frame** on a parameterized surface $x : U \to S$ is an orthonormal moving frame such that for each $x \in S$,

$$e_1(x), e_2(x) \in T_xS$$

- This implies $e_3(x)$ is normal to S
- In general, there is no adapted moving frame defined on all of a surface S