# MATH-GA1002 Multivariable Analysis

## Integral of $n$-Form on Oriented Rectangle
## Integral of 1-Form on Oriented Line Segment
## Integral of 1-Form on Oriented Parameterized Curve
## Outer Orientation of Boundary of Rectangle
## Fundamental Theorem of Calculus on Rectangle
## Exterior Derivative of 1-Form

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# Integration of $n$-form on $n$-Rectangle with Standard Orientation

- Let $R = [a^1, b^1] \times \cdots [a^n, b^n] \subset \mathbb{R}^n$ with coordinates $(x^1, \ldots, x^n)$
- Let $\omega = f\, dx^1 \wedge \cdots \wedge dx^n$ be a differential form on $R$
- The integral of $\omega$ over $R$ is defined to be

$$\int_R \omega = \int_R f = \int_{x^1=a^1}^{x^1=b^1} \cdots \int_{x^n=a^n}^{x^n=b^n} f(x^1, \ldots, x^n)\, dx^n \cdots dx^1$$

- Order matters!

# Integration of $n$-Form on Oriented Rectangle

- Let

$$R = \{(y^1, \dots, y^n) \in \mathbb{R}^2 \;:\; a^1 \leq y^1 \leq b^1, \dots, a^n \leq y^n \leq b^n\}$$

and given $\sigma \in S_n$,

$$\omega = f\, dy^{\sigma(1)} \wedge \cdots dy^{\sigma(n)}$$

- Then

$$\begin{aligned} \int_R \omega &= \int_R f(y)\, dy^{\sigma(1)} \wedge \cdots dy^{\sigma(n)} \\ &= \int_R f(y)\epsilon(\sigma)\, dy^1 \wedge \cdots \wedge dy^n \\ &= \int_{y^n=a^n}^{y^n=b^n} \cdots \int_{y^1=a^1}^{y^1=b^1} f(y^1, \dots, y^n)\, dy^1 \cdots dy^n \end{aligned}$$

# Integration of 1-form on Oriented Line Segment

- A 1-rectangle $R$ is a line segment $[a, b] \subset \mathbb{R}$, where $a < b$
- The integral of a 1-form $f\,dx$ on $R$ with the standard orientation is
$$\int_R f\,dx = \int_{x=a}^{x=b} f(x)\,dx$$
- The integral of a 1-form $f\,dx$ on $R$ with the orientation implied by $-dx$ is
$$\begin{aligned}\int_R f\,dx &= \int_R (-f)\,(-dx) \\ &= \int_{x=a}^{x=b} -f(x)\,dx\end{aligned}$$

# Integral over Oriented Line Segment

- The standard convention is that if $a < b$, then

$$\int_{x=a}^{x=b} f(x)\,dx$$

is

$$\int_R f\,dx$$

using the standard orientation and

$$\int_{x=b}^{x=a} f(x)\,dx$$

is

$$\int_R f\,dx$$

using the opposite orientation

# Integral Over Oriented Parameterized Curve

- Let $c : [a, b] \to \mathbb{R}^n$ be a smooth map
  - Denote the input parameter by $t \in [a, b]$
  - We do not assume that $a \leq b$
- Let $\theta$ be a 1-form on an open set that contains $c([a, b])$
- The integral of $\theta$ along the parameterized curve $c$ is defined to be

$$\int_c \theta = \int_{t=a}^{t=b} c^*\theta$$

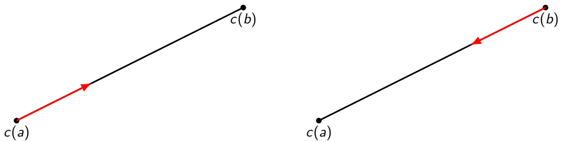
- If $c(t) = (x(t), y(t))$ and $\theta = f(x, y)\,dx + g(x, y)\,dy$, then

$$c^*dx = x'(t)\,dt$$
$$c^*dy = y'(t)\,dt$$

  and therefore

$$c^*\theta = (f(x(t), y(t))x'(t) + g(x(t), y(t))y'(t))\,dt$$

# Orientation of Curve




- Let $a < b$
- $C \subset \mathbb{R}^n$ is a nonempty connected **smooth curve** if there exists a smooth map

$$c : [a, b] \to \mathbb{R}^n$$

  such that
  - For each $t \in [a, b]$, $c'(t) \neq 0$
  - $c$ restricted to $(a, b)$ is injective
- An orientation of $C$ is the choice of a direction along the curve
- The orientation is in the direction of either $c'(t)$ or $-c'(t)$

# Integral of 1-form on Oriented Curve

- Let $O \subset \mathbb{R}^n$ be open, $a < b$, and $C \subset \mathbb{R}^n$ be a curve with a parameterization
$$c : [a, b] \to O$$
- Let $\theta = f_1\, dx^1 + \cdots + f_n\, dx^n$ be a 1-form on $O$
- The pullback of $\theta$ by $c$ is
$$\begin{aligned} c^*\theta &= f_1(c(t))(x^1)'(t)\, dt + \cdots + f_n(c(t))(x^n)'(t)\, dt \\ &= (f_1(c(t))(x^1)'(t) + \cdots + f_n(c(t))(x^n)'(t))\, dt \end{aligned}$$
- If the orientation of $C$ is $c'(t)$, then
$$\int_C \theta = \int_{[a,b]} c^*\theta = \int_{t=a}^{t=b} f_k(c(t))(x^k)'(t)\, dt$$
- If the orientation of $C$ is $-c'(t)$, then
$$\begin{aligned} \int_C \theta = \int_{[b,a]} c^*\theta &= \int_{t=b}^{t=a} f_k(c(t))(x^k)'(t)\, dt \\ &= -\int_{t=a}^{t=b} f_k(c(t))(x^k)'(t)\, dt \end{aligned}$$

# Integration of 2-form on Standard Rectangle

- Let $R = [a^1, b^2] \times [a^2, b^2] \subset \mathbb{R}^2$
- The integral of a 2-form is $f\, dx^1 \wedge dx^2$ on $R$ is

$$\int_R f\, dx^1 \wedge dx^2 = \int_{x^2=a^2}^{x^2=b^2} \int_{x^1=a^1}^{x^1=b^1} f(x^1, x^2)\, dx^1\, dx^2$$

- Example: If $R = [0,1] \times [0,1]$ and $\theta = x^1\, dx^2 \wedge dx^1$, then

$$\begin{aligned}
\int_R \theta &= \int_R x^1\, dx^2 \wedge dx^1 \\
&= \int_R -x^1\, dx^1 \wedge dx^2 \\
&= \int_{x^1=0}^{x^1=1} \int_{x^2=0}^{x^2=1} -x^1\, dx^2\, dx^1 \\
&= \int_{x^1=0}^{x^1=1} -x^1\, dx^1 \\
&= -\frac{1}{2}
\end{aligned}$$

# Example (Part 1)

- Consider
$$\int_R \omega,$$
where
$$R = \{(r,\theta) \in \mathbb{R}^2 \ : \ 0 \le r \le 1 \text{ and } 0 \le \theta \le \pi\}$$
and
$$\omega = r\, d\theta \wedge dr$$
- Here, there is no standard orientation

# Example (Part 2)

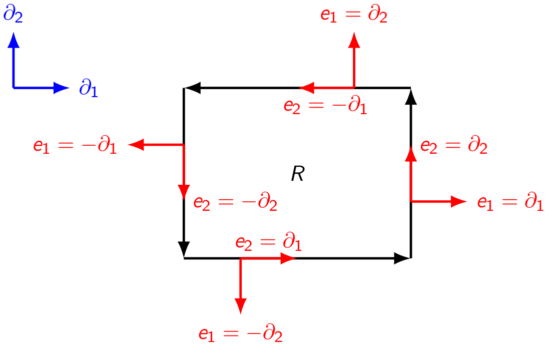
- If we use the orientation given by $dr \wedge d\theta$, then

$$\begin{aligned}
\int_R \omega &= \int_R -r\,dr \wedge d\theta \\
&= -\int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} r\,dr\,d\theta \\
&= -\int_{r=0}^{r=1} 2\pi r\,dr \\
&= -\pi
\end{aligned}$$

- If we use the orientation given by $d\theta \wedge dr$, then

$$\begin{aligned}
\int_R \omega &= \int_R r\,d\theta \wedge dr \\
&= \int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} r\,dr\,d\theta \\
&= \pi
\end{aligned}$$

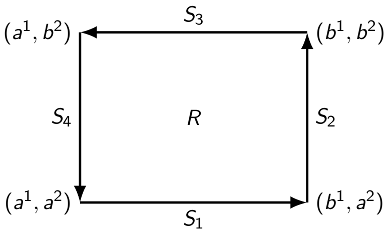
# Outer Orientation of Boundary of Rectangle




- Let $R \subset \mathbb{R}^2$ have the standard orientation given by the 1-form $dx^1 \wedge dx^2$ and standard basis $(\partial_1, \partial_2)$
- The **outer orientation** of each line segment on the boundary of $R$ is given by $e_2$, where $(e_1, e_2)$ is a positively oriented basis of $\mathbb{R}^2$, i.e.,

$$(dx^1 \wedge dx^2)(e_1, e_2) > 0$$

# Oriented Parameterization of Boundary




- $\partial R = S_1 \cup S_2 \cup S_3 \cup S_4$, where each side is parameterized by

$$
\begin{aligned}
S_1 &: \ \gamma_1(t) = (t, a^2), \ a^1 \leq t \leq b^1 \\
S_2 &: \ \gamma_2(t) = (b^1, t), \ a^2 \leq t \leq b^2 \\
S_3 &: \ \gamma_2(t) = (t, b^2), \ b^1 \geq t \geq a^1 \\
S_4 &: \ \gamma_3(t) = (a^1, t), \ b^2 \geq t \leq a^2
\end{aligned}
$$

# Integral of 1-Form on Oriented Boundary (Part 1)

- Let $\partial R$ denote the boundary of $R$ with the outward orientation
- Let $\theta = f_1\,dx^1 + f_2\,dx^2$ be a 1-form on $R$
- The integral of $\theta$ on $\partial R$ is defined to be

$$\begin{aligned}\int_{\partial R} \theta &= \int_{S_1} \theta + \int_{S_2} \theta + \int_{S_3} \theta + \int_{S_4} \theta \\ &= \int_{t=a^1}^{t=b^1} \gamma_1^*\theta + \int_{t=a^2}^{t=b^2} \gamma_2^*\theta + \int_{t=b^1}^{t=a^1} \gamma_3^*\theta + \int_{t=b^2}^{t=a^2} \gamma_4^*\theta\end{aligned}$$

# Integral of 1-Form on Oriented Boundary (Part 2)

- The pullbacks of the parameterized curves are

$$\gamma_1^*\theta = \gamma_1^*(f_1\,dx^1 + f_2\,dx^2) = f_1(t, a^2)\,dt$$
$$\gamma_2^*\theta = \gamma_2^*(f_1\,dx^1 + f_2\,dx^2) = f_2(b^1, t)\,dt$$
$$\gamma_3^*\theta = \gamma_3^*(f_1\,dx^1 + f_2\,dx^2) = f(t, b^2)\,dt$$
$$\gamma_4^*\theta = \gamma_4^*(f_1\,dx^1 + f_2\,dx^2) = f_2(a^1, t)\,dt$$

- Therefore,

$$\int_{\partial R}\theta$$
$$= \int_{t=a^1}^{t=b^1} f_1(t, a^2)\,dt + \int_{t=a^2}^{t=b^2} f_2(b^1, t)\,dt$$
$$+ \int_{t=b^1}^{t=a^1} f_1(t, b^2)\,dt + \int_{t=b^2}^{t=a^2} f_2(a^1, t)\,dt$$
$$= \int_{t=a^2}^{t=b^2} f_2(b^1, t) - f_2(a^1, t)\,dt - \int_{t=a^1}^{t=b^1} f_1(t, b^2) - f_1(t, a^2)\,dt$$

# Fundamental Theorem of Calculus on Rectangle

Using the standard orientation on $\mathbb{R}^2$,

$$\begin{aligned}
&\int_{\partial R} \theta \\
&= \int_{x^2=a^2}^{x^2=b^2} f_2(b^1, x^2) - f_2(a^1, x^2)\, dx^2 \\
&\quad - \int_{x^1=a^1}^{x^1=b^1} f_1(x^1, b^2) - f_1(x^1, a^2)\, dx^1 \\
&= \int_{x^2=a^2}^{x^2=b^2} \int_{x^1=a^1}^{x^1=b^1} \partial_1 f_2(x^1, x^2)\, dx^1\, dx^2 \\
&\quad - \int_{x^1=a^1}^{x^1=b^1} \int_{x^2=a^2}^{x^2=b^2} \partial_2 f_1(x^1, x^2)\, dx^2\, dx^1 \\
&= \int_{x^2=a^2}^{x^2=b^2} \int_{x^1=a^1}^{x^1=b^1} (\partial_1 f_2(x^1, x^2)\, dx^1\, dx^2 - \partial_2 f_1(x^1, x^2))\, dx^1\, dx^2 \\
&= \int_R (\partial_1 f_2 - \partial_2 f_1)\, dx^1 \wedge dx^2,
\end{aligned}$$

# Exterior Derivative of 1-Form

- Given a 1-form $\theta = f_1\,dx^1 + f_2\,dx^2$ on $R$, define its **exterior derivative** to be the 2-form

$$\begin{aligned} d\theta &= df_1 \wedge dx^1 + df_2 \wedge dx^2 \\ &= (\partial_1 f_1\,dx^1 + \partial_2 f_1\,dx^2) \wedge dx^1 + (\partial_1 f_2\,dx^1 + \partial_2 f_2\,dx^2) \wedge dx^2 \\ &= (\partial_1 f_2 - \partial_2 f_1)\,dx^1 \wedge dx^2 \end{aligned}$$

- The fundamental theorem of calculus on a 2-dimensional rectangle is

$$\int_{\partial R} \theta = \int_R d\theta$$

or

$$\int_{\partial R} P\,dx + Q\,dy = \int_R (\partial_x Q - \partial_y P)\,dx \wedge dy,$$

which is also known as Green's Theorem (on a rectangle)

# Basic Facts of Exterior Differentiation

- If $\theta^1, \theta^2$ are 1-forms, then

$$d(\theta^1 + \theta^2) = d\theta^1 + d\theta^2$$

- If $f$ is a scalar function and $\theta$ is a 1-form

$$d(f\theta) = df \wedge \theta + f\, d\theta$$

- If $f$ is a scalar function, then

$$d(df) = 0$$

- Given a 1-form $\theta$ on an open $P \subset \mathbb{R}^n$ and a map $F : O \to P$, where $O \subset \mathbb{R}^m$ is open,

$$d(F^*\theta) = F^*(d\theta)$$

- Also, recall that if $f : P \to \mathbb{R}$ is a smooth function, then

$$d(F^*f) = d(f \circ F) = F^*df$$