MATH-GA1002 Multivariable Analysis

Change of Variables Formula for Integral
Tangent Bundle
Pushforward of Tangent Vector
Cotangent Bundle
Differential 1-Forms
Pullback of Differential 1-Form

Deane Yang

Courant Institute of Mathematical Sciences
New York University

April 27, 2024

Change of Variables Formula for Single and Double Integrals

- The change of variable formula for a single integral is also known as substitution,

$$\int_{u=u(a)}^{u=u(b)} f(u)\, du = \int_{x=a}^{x=b} f(u(x))u'(x)\, dx.$$

- The change of variable formula for a double integral is more complicated:

$$\int_{x=\dots}^{x=\dots} \int_{y=\dots}^{y=\dots\,)} f(x,y)\, dy\, dx$$
$$= \int_{u=\dots}^{u=\dots} \int_{v=\dots}^{v=\dots} f(x(u,v), y(u,v))|\det J(u,v)|\, dv\, du,$$

where

$$J(u,v) = \begin{bmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{bmatrix}$$

Change of Variables Formula for Multiple Integral

- The change of variable formula for an m-dimensional integral is:

$$\int_{x^1=\dots}^{x^1=\dots} \cdots \int_{x^m=\dots}^{x^m=\dots)} f(x^1, \dots, x^m)\, dx^m \cdots\, dx^1$$
$$= \int_{u^1=\dots}^{u^1=\dots} \cdots \int_{u^m=\dots}^{u^m=\dots} f(x^1(u), \dots, x^m(u)) |\det J(u)|\, du^m \cdots du^1,$$

where

$$J(u) = \begin{bmatrix} \partial_1 x^1 & \cdots & \partial_m x^1 \\ \vdots & & \vdots \\ \partial_1 x^m & \cdots & \partial_m x^m \end{bmatrix}$$

Inconsistency Between Single and Double Integrals

- If $f(x, y)$ is always positive on a rectangle $R = [a, b] \times [c, d]$, then
$$\int_R f = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x, y)\, dy\, dx > 0$$
- If $f(x)$ is always positive on an interval $[a, b]$, then
$$\int_{x=a}^{x=b} f(x)\, dx = -\int_{x=b}^{x=a} f(x)\, dx$$
- For a single integral, the direction of integration matters
- We do this, so that the Fundamental Theorem of Calculus,
$$\int_{x=a}^{x=b} f'(x)\, dx = f(b) - f(a)$$
holds, **even if** $a > b$

Tangent Vectors

- Let $O \subset \mathbb{R}^m$ be open
- Recall that a tangent vector at $x \in O$ is the velocity vector of a parameterized curve
- Let T_xO be the space of all tangent vectors at x
- Given a tangent vector $v \in T_xO$, there is a smooth curve $c : I \to O$ such that

$$c(0) = x \text{ and } c'(0) = v$$

- Recall also that a tangent vector defines a directional derivative of functions
- Given a tangent vector $v \in \mathbb{R}^m$ at $x \in O$ and a smooth function $f : O \to \mathbb{R}$,

$$D_v f(x) = \left.\frac{d}{dt}\right|_{t=0} f(c(t))$$

Coordinate Tangent Vectors

- Given $x = (x^1, \dots, x^m) \in O$ and $1 \leq k \leq m$, let $c_k : I \to O$ be the curve such that

$$c_k(t) = (c_k^1(t), \dots, c_k^m(t)),$$

where for each $1 \leq j \leq m$,

$$c_k^j(t) = \begin{cases} x^j & \text{if } j \neq k \\ t & \text{if } j = k \end{cases}$$

- For each $1 \leq k \leq m$, let $\partial_k \in T_x O$ be the velocity vector
- Then $c_k(0) = x$ and

$$\partial_k = c_k'(0)$$

Tangent Space is a Vector Space

- Let $v \in T_x O$ and $c : I \to O$ be a curve such that $c(0) = x$ and $c'(0) = v$
- If $c(t) = (c^1(t), \ldots, c^m(t))$, then
$$v = c'(0) = ((c^1)'(0), \ldots, (c^m)'(0)) \in \mathbb{R}^m$$
- For any $a \in \mathbb{R}$, the curve $\tilde{c} : I \to O$ given by
$$\tilde{c}(t) = x + t(av)$$
has velocity $\tilde{c}'(0) = av$
- Given $x \in O$ and $v_1, v_2 \in T_x O$, if c_1 and c_2 are the curves given by
$$c_1(t) = x + tv_1 \text{ and } c_2(t) = x + tv_2,$$
then the curve $c : I \to O$ given by
$$c(t) = x + t(v_1 + v_2)$$
satisfies $c(0) = x$ and $c'(0) = v_1 + v_2$
- $T_p O$ satisfies all of the properties of a vector space

Tangent Bundle

- We will let

$$T_* O = \coprod_{x \in O} T_x O,$$

which is called the **tangent bundle** of O

Coordinate Tangent Vectors are a Basis of Tangent Space

- Any $v \in T_x O$ can be written with respect the coordinate tangent vectors $(\partial_1, \ldots, \partial_m)$ as

$$v = \partial_k a^k = a^1 \partial_1 + \cdots + a^m \partial_m$$

- I.e., there is a linear isomorphism

$$\begin{aligned} \mathbb{R}^m &\to T_x O \\ (a^1, \ldots, a^m) &\mapsto a^1 \partial_1 + \cdots + a^m \partial_m \end{aligned}$$

Pushforward of a Tangent Vector

- Let $U \subset \mathbb{R}^k$ and $F : U \to O$ be a smooth map
- Let $t = (t^1, \dots, t^k)$ denote the coordinates on U
- Recall that the **differential** of F at $t \in U$ is the linear map

$$\begin{aligned} \partial F(t) : T_t U &\to T_{F(t)} O \\ \tau &\mapsto D_\tau F(x) = \left.\frac{d}{dt}\right|_{t=0} F(c(t)), \end{aligned}$$

  where $c(0) = t$ and $c'(0) = \tau$
- The tangent vector $\partial F(x)(\tau)$ is called the **pushforward** of τ by F
- The pushforward map at x is also denoted

$$F_* : T_t U \to T_{F(t)} O$$

Cotangent Space and Bundle

- For each $x \in O$, denote $T_x^* O = (T_x O)^*$
- And denote
$$T^* O = \coprod_{x \in O} T_x^* O$$

Differentials of Functions

- Let $O \subset \mathbb{R}^m$ be open and $f : O \to \mathbb{R}$ be a smooth function
- Recall that the differential of f at each $x \in O$ is a linear map

$$df(x) : T_x O \to \mathbb{R},$$

where, for each $v \in T_x O$,

$$\langle df(x), v \rangle = D_v f(x) = \left. \frac{d}{dt} \right|_{t=0} f(c(t)),$$

where $c : I \to O$ is a smooth curve such that $c(0) = x$ and $c'(0) = v$

Differentials of Coordinate Functions

- Let $(\partial_1, \ldots, \partial_m)$ denote the standard basis of $\mathbb{R}^m$
- For each $1 \leq k \leq m$, there is the coordinate function
$$x^k : \mathbb{R}^m \to \mathbb{R}$$
$$(a^1, \ldots, a^m) \mapsto a^k$$
- Let $v = \partial_k v^k \in T_x O$
- Let$c : I \to O$ be a curve such that $c'(0) = v$
- Then, if $c = (c^1, \ldots, c^m)$, then for any $1 \leq j \leq m$, differential of x^j at $x \in O$ is $dx^j \in T_x^* O$, where
$$\langle dx^j(x), v \rangle = \left. \frac{d}{dt} \right|_{t=0} x^j(c(t)) = (c^j)'(0) = v^j$$
- In particular,
$$\langle dx^j, \partial_k \rangle = \delta^j_k$$
- Therefore, $(dx^1, \ldots, dx^m)$ is the basis of $T_x^* O$ that is dual to the standard basis $(\partial_1, \ldots, \partial_m)$ of $T_x O$

Differential of Function With Respect to Coordinates

- Recall that if $f : O \to \mathbb{R}$ is a smooth function, then its differential is given by

$$df = \partial_k f \, dx^k,$$

where for each $x \in O$, $df(x) \in T_x O$

- For any $v = \partial_k v^k \in T_x O$, let $c : I \to O$ be a curve such that $c(0) = x$ and $c'(0) = v$
- Then, by the chain rule,

$$\begin{aligned}
\langle df(x), v \rangle &= D_v f(x) \\
&= \left.\frac{d}{dt}\right|_{t=0} f(c(t)) \\
&= \partial_k f(c(0))(c^k)'(0) \\
&= \partial_k f(x) v^k \\
&= \partial_k f(x) \langle dx^k, v \rangle
\end{aligned}$$

Differential 1-Forms

- A **differential** 1-**form** or just 1-**form** is a map

$$\theta : O \to T^*O,$$

  such that $\theta(x) \in T_x^*O$

- I.e., for each x, $\theta(x)$ is a linear function on $\mathbb{R}^m$,

$$\begin{aligned} \theta(x) : \mathbb{R}^m &\to \mathbb{R} \\ v &\mapsto \langle \theta(x), v \rangle \end{aligned}$$

- Since $(dx^1, \dots, dx^m)$ is a basis of $(\mathbb{R}^m)^*$, it follows that for each $x \in O$, there exist coefficients $a_1(x), \dots, a_m(x)$ such that

$$\theta(x) = a_1(x)\, dx^1 + \cdots + a_m(x)\, dx^m$$

- θ is a **smooth** 1-form if the functions $a_1, \dots, a_m$ are smooth

Pullback of 1-Form

- Recall that a linear map $L : V \to W$ induces a natural map

$$L^* : W^* \to V^*,$$

where for each $\beta \in W^*$, $L^*\beta = \beta \circ L \in V^*$, i.e., for each $v \in V$,

$$L^*\beta(v) = \beta(L(v))$$

- $L^*\beta$ is called the **pullback** of *beta* by L
- Let $O \subset \mathbb{R}^m$ and $P \subset \mathbb{R}^n$ be open
- Let $F : O \to P$ be a smooth map
- The pushforward map at $x \in O$ is a linear map

$$F_* : \mathbb{R}^m \to \mathbb{R}^n$$

- The **pullback** by F is a linear map

$$F^* : (\mathbb{R}^n)^* \to (\mathbb{R}^m)^*$$

Pullback of 1-Form (Part 2)

- For any 1-form β on P and $y \in P$, $\beta(y) \in (\mathbb{R}^n)^*$ is a linear function
$$\beta(y) : T_y P \to \mathbb{R}$$
- By definition, the pullback $F^*\beta$ at $x \in O$ is the linear function
$$(F^*\beta)(x) : \mathbb{R}^m \to \mathbb{R}$$
given by $(F^*\beta)(x) = \beta(F(x)) \circ F_*$
- I.e., given a tangent vector $v \in \mathbb{R}^m$ at $x \in \mathbb{R}^m$,
$$\begin{aligned} \langle (F^*\beta)(x), v \rangle &= \langle \beta(F(x)) \circ F_* \rangle \\ &= \langle \beta(F(x)), F_* v \rangle \end{aligned}$$

Pullback of Differential of Function

- Let $f : P \to \mathbb{R}$ be a smooth function and $F : O \to P$ be a smooth map
- Recall that for any $v \in T_x O$, and $c : I \to O$ such that $c(0) = x$ and $c'(0) = v$,

$$
\begin{aligned}
D_v(f \circ F)(x) &= \langle d(f \circ F)(x), v \rangle \\
&= \left.\frac{d}{dt}\right|_{t=0} f(F(c(t))) \\
&= \left.\frac{d}{dt}\right|_{t=0} f((F\circ)(t))) \\
&= \langle df(x), (F \circ c)'(0) \rangle \\
&= \langle df(x), F_* v \rangle \\
&= \langle (F^* df)(x), v \rangle
\end{aligned}
$$

Coordinate Vectors and 1-Forms

- Let $(x^1, \ldots, x^m)$ denote coordinates on O
- Let $(y^1, \ldots, y^m)$ denote coordinates on P
- Denote the coordinate basis of $T_x O$ by

$$(\partial_1^x, \ldots, \partial_m^x) = \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m} \right)$$

- Denote the coordinate basis of $T_y P$ by

$$(\partial_1^y, \ldots, \partial_m^y) = \left(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^m} \right)$$

- Each coordinate on P can be viewed as a function

$$y^a : P \to \mathbb{R}$$

- The differentials $(dy^1, \ldots, dy^n)$ are the basis of $T_y^* P$ dual to the coordinate basis of $T_y P$

Pullback of Coordinate 1-Form

- For each $x \in O$, denote
$$F(x) = (y^1(x), \ldots, y^n(x)) \in P$$
- For each $1 \le a \le n$, denote
$$(y^a \circ F)(x) = y^a(x)$$
- Then, for each $v \in T_x O$ and $1 \le a \le n$,
$$\begin{aligned}\langle (F^* dy^a)(x), v\rangle &= \langle d(y^a \circ F), v\rangle \\ &= \langle dy^a, v\rangle\end{aligned}$$
- With respect to coordinates on O, if $v = v^i \partial_i^x \in T_x O$, then
$$\begin{aligned}\langle (F^* dy^a)(x), v\rangle &= \langle dy^a, v\rangle \\ &= \langle \partial_i^x y^a \, dx^i, v^j \partial_j^x\rangle \\ &= v^j \frac{\partial y^a}{\partial x^j}\end{aligned}$$

Pullback of 1-Form in Coordinates

- $F^* dy^a = \partial_j y^a \, dx^j$
- The pullback by F of the 1-form

$$\beta = b_a \, dy^a$$

is

$$F^* \beta = b_a \frac{\partial y^a}{\partial x^j} \, dx^j$$

- Here, $b_a = b_a \circ F$ and $y^a = y^a \circ F$