MATH-GA1002 Multivariable Analysis

Area of Parallelogram
Oriented Area
Permutations
Sign of Permutation
Exterior m-Tensors
Orientation of Vector Space

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Affine Transformations and Parallelograms

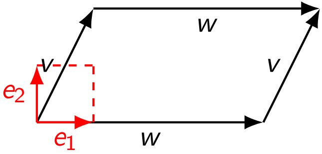
- An **affine transformation** is a map $A : \mathbb{R}^n \to \mathbb{R}^n$, where there exists a linear isomorphism $L : \mathbb{R}^n \to \mathbb{R}^n$ and $\tau \in \mathbb{R}^n$ such that

$$A(x) = \tau + L(x),$$

- A **parallelogram** is an affine transformation of a rectangle, i.e., P is a parallelogram if there exists a rectangle R and an affine transformation A such that

$$P = A(R)$$

Parallelogram in Vector Space



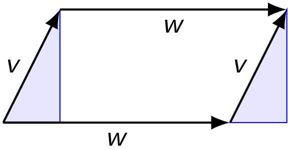
- Let $P(v, w)$ be the parallelogram with sides $v, w \in \mathbb{R}^2$.

$$P(v, w) = \{av + bw \ : \ 0 \leq a, b \leq 1\}.$$

- Since each side has measure 0, we can define the area of the parallelogram to be

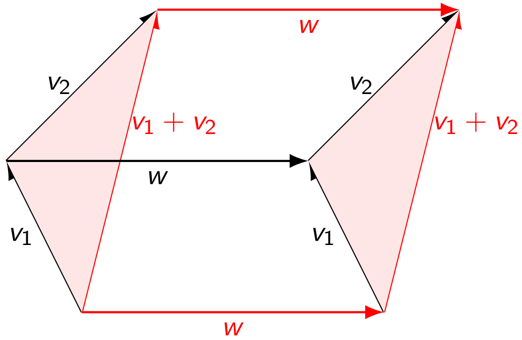
$$A(v, w) = \int \chi_{P(v,w)}$$

Area of Parallelogram



- Any parallelogram $P(v, w)$ can be decomposed into a rectangle and two congruent triangles
- The parallelogram has the same area as the rectangle with the same base and height

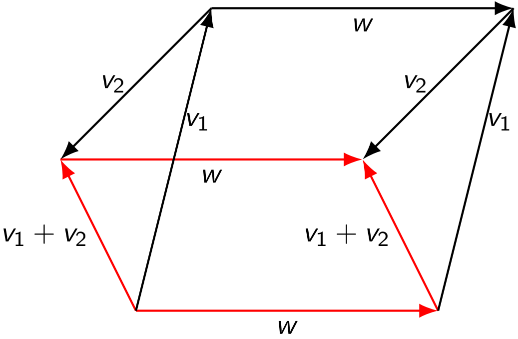
Area of Two Parallelograms with Parallel Bases



- If v_1 and v_2 both point upward relative to w, then

$$A(v_1 + v_2, w) = A(v_1, w) + A(v_2, w)$$

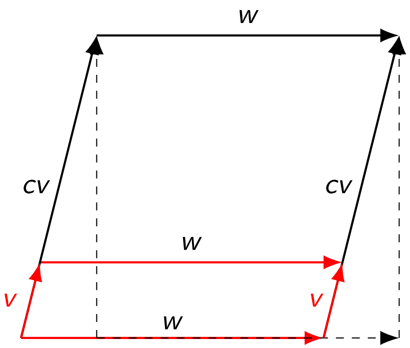
Area of Two Parallelograms with Parallel Bases



- If v_1 points upward and v_2 points downward relative to w, then

$$A(v_1 + v_2, w) = A(v_1, w) - A(v_2, w)$$

Area of rescaled parallelogram



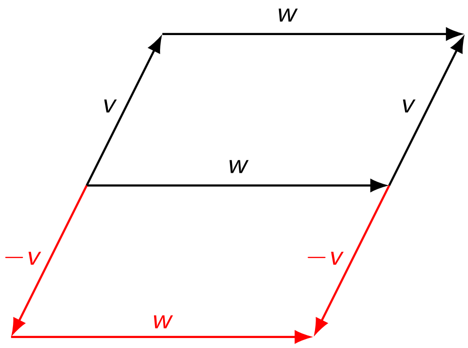
- If $c \geq 0$,

$$A(cv, w) = cA(v, w)$$

- In general,

$$A(cv, w) = |c|A(v, w)$$

Area of reflected parallelogram



$$A(-v, w) = A(v, w)$$

Area Versus Oriented Area

- The area function is awkward to use
  - $A(tv, w) = |t|A(v, w)$
  - It is not a differentiable function of v and w
- Redefine A so that it is bilinear and therefore sometimes negative
- Since $A(v, v) = 0$, it is alternating

$$\begin{aligned} 0 &= A(v + w, v + w) \\ &= A(v, v) + A(v, w) + A(w, v) + A(w, w) \\ &= A(v, w) + A(w, v) \end{aligned}$$

- $A(v, w)$ is called the *oriented area* of $P(v, w)$

Oriented Area of Parallelogram

- A is an exterior 2-tensor:

$$A(v_1 + v_2, w) = A(v_1, w) + A(v_2, w)$$
$$A(cv, w) = cA(v, w)$$
$$A(w, v) = -A(v, w)$$

- Since $A(e_1, e_2) = 1$ is a positively oriented orthonormal basis, if $v_1 = a_1^1 e_1 + a_1^2 e_2$ and $v_2 = a_2^1 e_1 + a_2^2 e_2$, then

$$\begin{aligned} A(v_1, w_2) &= A(a_1^1 e_1 + a_1^2 e_2, a_2^1 e_1 + a_2^2 e_2) \\ &= (a_1^1 a_2^2 - a_1^2 a_2^1) \\ &= \det \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix} \end{aligned}$$

- The sign of $A(v_1, v_2)$ depends on determinant of the change of basis matrix

Generalization to Higher Dimensions

- Let V be an m-dimensional vector space
- An m-tensor

$$A : V \times \cdots \times V \to \mathbb{R}$$

- In $\mathbb{R}^m$, the **parallelopiped** spanned by vectors $(v_1, \ldots, v_m)$ is defined to be

$$P(v_1, \ldots, v_m) = \{a^1 v_1 + \cdots a^m v_m \ : \ 0 \le a^1, \ldots, a^m \le 1\}$$

- If $(e_1, \ldots, e_m)$ is the standard basis of $\mathbb{R}^m$, then

$$P(e_1, \ldots, e_m) = 1$$

  is a rectangle (in fact, a cube)
- Its volume is defined to be

$$\text{vol}(e_1, \ldots, e_m) = 1, \qquad (1)$$

Permutations

- A **permutation of order** m is a bijective map

$$\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$$

- A permutation defines an ordered set

$$(\sigma(1), \ldots, \sigma(m)),$$

  where each integer appears exactly once
- Let S_m denote the set of all permutations of order m
- S_m is a subset of the space $\mathcal{M}_m$ of all maps from $\{1, \ldots, m\}$ to itself,

Permutations Comprise a Group

- Group multiplication is composition of maps
- For any $\sigma_1, \sigma_2 \in S_m$,

$$\sigma_2 \circ \sigma_1 \in S_m$$

- For any $\sigma_1, \sigma_2, \sigma_3 \in S_m$

$$\sigma_3 \circ (\sigma_2 \circ \sigma_1) = (\sigma_3 \circ \sigma_2) \circ \sigma_1$$

- There exists a unique permutation $e \in S_m$ such that for any $k \in \{1, \ldots, m\}$,

$$e(k) = k$$

- For any $\sigma \in S_m$,

$$\sigma \circ e = e \circ \sigma = \sigma$$

- For any $\sigma \in S_m$, there exists a unique $\sigma^{-1} \in S_m$ such that

$$\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = e$$

Notation for Permutations

- Let

$$(a_1 \ a_2 \ \ldots \ a_k)$$

denote the permutation such that

$$\sigma(a_j) = a_{j+1} \text{ if } 1 \leq j \leq k-1$$
$$\sigma(a_k) = a_1$$
$$\sigma(i) = i \text{ if } i \notin \{a_1, \ldots, a_k\}$$

Sign of Permutation

- A **transposition** is an element $\tau \in S_m$ such that for some $j \neq k$,
$$\tau(j) = k$$
$$\tau(k) = j$$
$$\tau(i) = i \text{ if } i \neq j \text{ and } i \neq k$$
- There exists a unique function
$$\epsilon : S_m \to \{-1, 1\}$$
satisfying the following:
  - $\epsilon(e) = 1$
  - For any transposition τ,
$$\epsilon(\tau) = -1$$
  - For any $\sigma_1, \sigma_2 \in S_m$,
$$\epsilon(\sigma_2 \circ \sigma_1) = \epsilon(\sigma_2)\epsilon(\sigma_1)$$
- $\epsilon(\sigma)$ is called the **sign** of σ

Sign of Map

- This can be extended to a function

$$\epsilon : \mathcal{M}_m \to \{-1, 0, 1\},$$

where if $\sigma \in \mathcal{M}_m$ is not bijective, then

$$\epsilon(\sigma) = 0$$

Examples

- $\tau = (1\ 2) \in S_3$ is the permutation such that

$$\tau(1) = 2,\ \tau(2) = 1,\ \tau(3) = 3$$

- It is a transposition and therefore $\epsilon(\tau) = -1$
- $\sigma = (1\ 2\ 3) \in S_3$ satisfies

$$\sigma(1) = 2,\ \sigma(2) = 3,\ \sigma(3) = 1$$

- Since

$$(1\ 2\ 3) = (1\ 2) \circ (2\ 3),$$

  it follows that $\epsilon(\sigma) = 1$

Exterior m-Tensors on m-Dimensional Vector Space

- An m-tensor of an m-dimensional vector space V

$$T : V \times \cdots \times V \to \mathbb{R}$$

is **alternating** or **exterior** if for any $\{v_1, \ldots, v_m\} \subset V$ and transposition $\tau \in S_m$,

$$T(v_{\tau(1)}, \ldots, v_{\tau(m)}) = -T(v_1, \ldots, v_m)$$

- Equivalently, for any $\sigma \in S_m$,

$$T(v_{\sigma(1)}, \ldots, v_{\sigma(m)}) = \epsilon(\sigma) T(v_1, \ldots, v_m)$$

- The space of all alternating m-tensors will be denoted $\Lambda^m V^*$

$\dim(V) = m \implies \dim(\Lambda^m V^*) = 1$

- Let $T \in \Lambda^m V^*$
- Let $(e_1, \ldots, e_m)$ be a basis of V
- Let $v_k = e_1 a_k^1 + \cdots + e_m a_k^j \in V,\ 1 \leq k \leq m$
- Then

$$
\begin{aligned}
T(v_1, \ldots, v_m) &= T(e_{j_1} a_1^{j_1}, \ldots, e_{j_m} a_m^{j_m}) \\
&= \sum_{j_1=1}^{m} \cdots \sum_{j_m=1}^{m} T(e_{j_1} a_1^{j_1}, \ldots, e_{j_m} a_m^{j_m}) \\
&= \sum_{\sigma \in \mathcal{M}_m} a_1^{\sigma(1)} \cdots a_m^{\sigma(m)} T(e_{\sigma(1)}, \ldots, e_{\sigma(m)}) \\
&= \sum_{\sigma \in \mathcal{M}_m} a_1^{\sigma(1)} \cdots a_m^{\sigma(m)} \epsilon(\sigma) T(e_1, \ldots, e_m) \\
&= \left(\sum_{\sigma \in \mathcal{M}_m} \epsilon(\sigma) a_1^{\sigma(1)} \cdots a_m^{\sigma(m)} \right) T(e_1, \ldots, e_m) \\
&= (\det(A)) T(e_1, \ldots, e_m)
\end{aligned}
$$

Volume of Parallelopiped in $\mathbb{R}^m$

- Using geometric arguments as above, it can be shown that there is a unique exterior m-tensor

$$\mathrm{vol} : \mathbb{R}^m \times \cdots \times \mathbb{R}^m \to \mathbb{R},$$

  such that the n-dimensional volume of a parallelopiped $P(v_1, \ldots, v_m)$ is equal to

$$|\,\mathrm{vol}(v_1, \ldots, v_m)|$$

- We therefore defined the n-dimensional **oriented volume** of $P(v_1, \ldots, v_m)$ to be

$$\mathrm{vol}(v_1, \ldots, v_m)$$

- If $v_k = e_j a^j_k$, where $(e_1, \ldots, e_m)$ is the standard basis of $\mathbb{R}^m$, then

$$\mathrm{vol}(v_1, \ldots, v_m) = \det(A)$$

Orientation of a Basis in $\mathbb{R}^m$

- Let $(v_1, \ldots, v_m)$ be an ordered basis of $\mathbb{R}^m$
- The basis is **positively oriented** if

$$\mathrm{vol}(v_1, \ldots, v_m) > 0$$

- The order of the basis vectors matters!

Orientation of a Vector Space

- The space $\Lambda^m V^*$ of alternating m-tensors is 1-dimensional
- Therefore, if $A_1, A_2 \in \Lambda^m V^*$ are both nonzero, then there exists a nonzero $c \in \mathbb{R}$ such that $A_2 = cA_1$
- It follows that $\Lambda^m V^* \backslash \{0\}$ has two connected components, where $A_1, A_2 = cA_1$ lie in the same component if $c > 0$ and different components if $c < 0$
- Each component is called an **orientation** on V
- Any nonzero $\Theta \in \Lambda^m V^*$ determines an orientation
- An oriented vector space is a vector space with an orientation, denoted $\Lambda^m_+ V^*$, called the positive orientation
- An ordered basis $(v_1, \ldots, v_m)$ is **positively oriented** if for any $\Theta \in \Lambda^m_+ V^*$,

$$\Theta(v_1, \ldots, v_m) > 0$$