MATH-GA1002 Multivariable Analysis

Riemann Integration
Sets of Measure Zero
Sets of Content Zero
Fubini Theorem
Basic Properties of Integrals

Deane Yang

Courant Institute of Mathematical Sciences
New York University

March 30, 2024

Partition of an Interval

- A **partition of an interval** $[a, b]$ is a finite sequence

$$P = (t_0, \ldots, t_N),$$

  where

$$a = t_0 \leq t_1 \leq \cdots \leq t_N = b$$

- A partition

$$\widetilde{P} = (\tilde{t}_0, \ldots, \tilde{t}_{\widetilde{N}})$$

  is a **refinement** of P if

$$\{t_0, \ldots, t_N\} \subset \{\tilde{t}_0, \ldots, \tilde{t}_{\widetilde{N}}\}$$

Rectangles and Their Volumes

- A **rectangle** in $\mathbb{R}^n$ is a set of the form

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n],$$

  where $a_k \leq b_k$ for each $1 \leq k \leq n$
- The **volume** of R is

$$\text{vol}(R) = (b_1 - a_1) \cdots (b_n - a_n)$$

Partition of a Rectangle

- A **partition of a rectangle** R consists of n partitions,

$$P = (P_1, \ldots, P_n),$$

such that for each $1 \leq i \leq n$,

$$P_i = (t_{i,0}, \ldots, t_{i,N_i})$$

is a partition of $[a_i, b_i]$

- A partition of R subdivides R into $N = N_1 \cdots N_n$ rectangles, where for each

$$1 \leq i_1 \leq N_1,\ 1 \leq i_2 \leq N_2, \ldots, 1 \leq i_n \leq N_n,$$

there is the subrectangle

$$R_{i_1,i_2,\ldots,i_n} = [t_{1,i_1-1}, t_{1,i_1}] \times [t_{2,i_2-1}, t_{2,i_2}] \times \cdots \times [t_{n,i_n-1}, t_{n,i_n}]$$

Upper and Lower Riemann Sums

- Let $f : R \to \mathbb{R}$ be a bounded function
- Let P be a partition of R
- The volume of each subrectangle $S = R_{i_1, i_2, \ldots, i_n}$ is

$$\mathrm{vol}(S) = (t_{i_1} - t_{i_1 - 1})(t_{i_2} - t_{i_2 - 1}) \cdots (t_{i_n} - t_{i_n - 1})$$

- For each subrectangle S of P, let

$$m(f, S) = \inf\{f(x) \ : \ x \in S\} \text{ and } M(f, S) = \sup\{f(x) \ : \ x \in S\}$$

- The **lower Riemann sum** of f for P is

$$L(f, P) = \sum_S m(f, S)\, \mathrm{vol}(S)$$

- The **upper Riemann sum** of f for P is

$$L(f, P) = \sum_S m(f, S)\, \mathrm{vol}(S)$$

Refinement of Partition of Rectangle

- A partition
$$\widetilde{P} = (\widetilde{P}_1, \ldots, \widetilde{P}_n)$$
is a **refinement** of
$$P = (P_1, \ldots, P_n)$$
if each $\widetilde{P}_i$ is a refinement of P_i
- Each subrectangle S of P is a union of subrectangles of $\widetilde{P}$,
$$S = \widetilde{S}_1 \cup \cdots \widetilde{S}_{N(\widetilde{P},S)},$$
where $N(\widetilde{P}, S)$ is the number of subrectangles of $\widetilde{P}$ contained in S
- Also,
$$\operatorname{vol}(S) = \operatorname{vol}(\widetilde{S}_1) + \cdots + \operatorname{vol}(\widetilde{S}_{N(\widetilde{P},S)})$$

Refinements of Riemann Sums

- If $\widetilde{S} \subset S$, then

$$m(f, S) \leq m(f, \widetilde{S} \leq M(f, \widetilde{S}) \leq M(f, S)$$

- It follows that

$$\begin{aligned} L(f, P) &= \sum_S m(f, S) \operatorname{vol}(S) \\ &\leq \sum_S \sum_{\widetilde{S} \subset S} m(f, \widetilde{S}) \operatorname{vol}(\widetilde{S}) \\ &= \sum_{\widetilde{S}} m(f, \widetilde{S}) \operatorname{vol}(\widetilde{S}) \\ &= L(f, \widetilde{P} \end{aligned}$$

- Similarly,

$$U(f, P) \geq U(f, \widetilde{P})$$

- Therefore, if P' is a refinement of P, then

$$L(f, P) \leq L(f, \widetilde{P}) \leq U(f, \widetilde{P}) \leq U(f, P)$$

Riemann Integrable Functions

- Given a rectangle $R \subset \mathbb{R}^n$, a function $f : R \to \mathbb{R}$ is **Riemann integrable** if

$$\sup\{L(f,P) : \ P \text{ is a partition of } R\} \\ = \inf\{U(f,P) : \ P \text{ is a partition of } R\}$$

- The **integral** of a Riemann integrable function f over R is defined to be

$$\int_R f = \sup\{L(f,P) \ : \ P \text{ is a partition of } R\} \\ = \inf\{U(f,P) \ : \ P \text{ is a partition of } R\}$$

Sets of Measure Zero

- A subset $A \subset \mathbb{R}^n$ has **measure zero** if for any $\epsilon > 0$, there exists countably many rectangles $R_1, R_2, \ldots$ such that

  $$A \subset \bigcup_{i=1}^{\infty} R_i$$

  and

  $$\sum_{i=1}^{\infty} \text{vol}(R_i) \leq \epsilon$$

Examples of Sets of Measure Zero

- A finite set $A \subset \mathbb{R}^n$ has measure zero
- A countable set $A = \{a_1, a_2, \cdots\} \subset \mathbb{R}^n$ has measure zero
  - Because for any $\epsilon > 0$, if R_i is a rectangle such that $a_i \in R_i$ and

$$\text{vol}(R_i) = \epsilon 2^{-i},$$

then

$$A \subset \bigcup_{i=1}^{\infty} R_i$$

$$\sum_{i=1}^{\infty} \text{vol}(R_i) = \sum_{i=1}^{\infty} \epsilon 2^{-i} = \epsilon$$

- If $A_1, A_2, \ldots,$ is a countable collection of sets with measure zero, then their union

$$A = \bigcup_{i=1}^{\infty} A_i$$

has measure zero

Sets of Content Zero

- A subset $A \subset \mathbb{R}^n$ has **content zero** if for any $\epsilon > 0$, there exists a *finite* collection of rectangles $R_1, \ldots, R_N \subset \mathbb{R}^n$ such that

$$A \subset R_1 \cup \cdots \cup R_N \text{ and } \operatorname{vol}(R_1) + \cdots + \operatorname{vol}(R_N) < \epsilon.$$

- If a set has content zero, then it has measure zero

Compact and Measure Zero Implies Content Zero

- **Theorem:** If $A \subset \mathbb{R}^n$ is compact and has measure 0, then it has content 0
- Let $\epsilon > 0$. Since A has measuer 0, there exists a countable cover of A by rectangles, $\{R_1, \ldots, \}$ such that

$$\sum_{i=1}^{\infty} \text{vol}(R_i) < \epsilon$$

- **Proof:** Since A is compact, there exists a finite subcover $R_{i_1}, \ldots, R_{i_N}$ of A and

$$\sum_{j=1}^{N} \text{vol}(R_{i_j} \leq \sum_{i=1}^{\infty} \text{vol}(R_i) < \epsilon$$

Since this holds for any $\epsilon > 0$, A has content 0

Nonempty Closed Interval in $\mathbb{R}$ Does Not Have Measure 0

- Let $a < b$
- **Theorem:** The interval $[a, b] \subset \mathbb{R}$ does not have content zero
- **Proof:** Let $R_1, \ldots, R_N$ be rectangles, i.e., nonempty connected compact intervals, such that

$$[a, b] \subset R_1 \cup \cdots \cup R_N$$

- Let

$$a = t_0 < t_1 < \cdots < t_M = b$$

be all endpoints of all $R_1, \ldots, R_N$, listed in increasing order
- Since each $[t_{k-1}, t_k]$ lies in at least one of the R_i, it follows that

$$b - a = \sum_{k=1}^{M} t_k - t_{k-1} \leq \sum_{i=1}^{N} \text{vol}(R_i)$$

- **Corollary:** The interval $[a, b] \subset \mathbb{R}$ does not have measure zero

Integrable Functions on a Closed Rectangle

- Let $R \subset \mathbb{R}^n$ be a closed rectangle
- Let $f : R \to \mathbb{R}$ be a bounded function and

$$B = \{x \in R \ : \ f \text{ is not continuous at } x\}$$

- **Theorem:** f is integrable if and only if B has measure 0

Fubini Theorem

- Let $R = [a^1, b^1] \times \cdots \times [a^n, b^n]$
- Let $f : R \to \mathbb{R}$ be a function satisfying the following:
  - f is Riemann integrable on R
  - Given any lower dimensional rectangle $R' \subset R$, the restriction of f to R' is Riemann integrable
- Then integral can be calculated as a sequence of 1-variable integrals

$$\int_R f$$
$$= \int_{x^1=a^1}^{x^1=b^1} \left(\int_{x^2=a^2}^{x^2=b^2} \cdots \left(\int_{x^n=a^n}^{x^n=b^n} f(x^1, \dots, x^n)\, dx^n \right) \cdots dx^2 \right) dx^1$$

- Second, it does not matter which order you do the integrals in

Interior and Boundary of a set $A \subset \mathbb{R}^n$

- Let $A \subset \mathbb{R}^n$
- The **interior of** A is the maximal open subset of A, i.e., the set of all points $x \in A$ such that there exist an open neigborhood of x contained in A
- The **boundary of** A is the set of all points $x \in \mathbb{R}^n$ such that any open neigborhood of x contains at least one point in A and at least one point not in A

Integrability of Characteristic Function of a Subset

- The **characteristic** or **indicator** function of a subset $C \subset \mathbb{R}^n$ is defined by
$$\chi_C(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases}$$
- **Theorem:** χ_C is integrable if and only if the boundary of C has measure 0
- **Proof:** If x is in the interior of C, then there exists an open neighborhood $O \subset C$ of x and therefore χ_C continuous at x
- If x is in the exterior of C, then there exists an open neighborhood $O \subset \mathbb{R}^n \backslash C$ of x and therefore χ_C continuous at x
- If x is in the boundary of C, then any neighborhood of x contains both a point in C and a point not in C and therefore χ_C is not continuous at C
- It follows that χ_C is integrable if and only if the boundary of C has measure 0

Basic Properties of Volumes of Rectangles

- For any rectangle $R \subset \mathbb{R}^n$ and $\tau \in \mathbb{R}^n$, let

$$R + \tau = \{x + \tau \; : \; x \in R\}$$

- Then

$$\text{vol}(R + \tau) = \text{vol}(R)$$

- If D is a diagonal matrix with nonnegative diagonal values, then DR is also a rectangle and

$$\text{vol}(DR) = \det(D)\,\text{vol}(R)$$

- The boundary of a rectangle has measure 0 (and therefore content 0)
- If R_1, R_2 are rectangles with disjoint interiors such that $R_1 \cup R_2$ is a rectangle, then

$$\text{vol}(R_1 \cup R_2) = \text{vol}(R_1) + \text{vol}(R_2)$$

Basic Properties of Integrals over Rectangles

- If f is integrable on a rectangle $R \subset \mathbb{R}^n$ and $\tau \in \mathbb{R}^n$, then

$$\int_{R+\tau} f = \int_R f_\tau \, dx,$$

  where $f_\tau(x) = f(x + \tau)$
- If D is a diagonal matrix with nonnegative diagonal values, then

$$\int_{DR} f = \det(D) \int_R f_D,$$

  where $f_D(x) = f(Dx)$

Basic Properties of Volumes of Domains in $\mathbb{R}^n$

- Let $M \subset \mathbb{R}^n$ be a bounded set whose boundary has measure 0
- There exists a rectangle R such that M lies in its interior
- Define the volume of M to be

$$\mathrm{vol}(M) = \int_R \chi_M$$

- For any $\tau \in \mathbb{R}^n$, let

$$M + \tau = \{x + \tau \ : \ x \in M\}$$

- Then

$$\mathrm{vol}(M + \tau) = \mathrm{vol}(M)$$

- If D is a diagonal matrix with nonnegative diagonal values, then DM is also a domain whose boundary has measure 0 and

$$\mathrm{vol}(DR) = \det(D)\,\mathrm{vol}(R)$$

- If M_1, M_2 are domains whose boundaries have measure 0 and whose interiors are disjoint, then

$$\mathrm{vol}(M_1 \cup M_2) = \mathrm{vol}(M_1) + \mathrm{vol}(M_2)$$

Basic Properties of Integrals over a Domain

- Let M be a bounded domain whose boundary has measure 0
- A function $f : M \to \mathbb{R}$ is integrable if it is continuous except on a set of measure 0
- Define the integral of f over M to be
$$\int_M f = \int_R f \chi_M,$$
where R is a rectangle such that M lies in its interior
- For any $\tau \in \mathbb{R}^n$ and integrable $f : M + \tau \to \mathbb{R}$,
$$\int_{M+\tau} f = \int_M f_\tau \, dx,$$
where $f_\tau(x) = f(x + \tau)$
- If D is a diagonal matrix with nonnegative diagonal values and $f : DM \to \mathbb{R}$, then
$$\int_{DR} f = \det(D) \int_R f_D,$$
where $f_D(x) = f(Dx)$