MATH-GA1002 Multivariable Analysis

Inverse Function Theorem

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Linear Isomorphism Implies Local Nonlinear Isomorphism

- Let V and W be n-dimensional vector spaces
- If $L : V \to W$ is a linear map with maximal rank, then
  - L is bijective
  - Its inverse map $L^{-1} : W \to V$ is linear
  - I.e., L and L^{-1} are linear isomorphisms
- Let $O \subset \mathbb{R}^n$ be open and $\Phi : O \to \mathbb{R}^n$ be a C^1 map
- Suppose that at $x_0 \in O$, the differential of Φ

  $$\partial\Phi(x_0) : \mathbb{R}^n \to \mathbb{R}^n$$

  is a linear isomorphism
- Then there exists an open neighborhood $N \subset O$ of x_0 such that

  $$\Phi|_N : N \to \Phi(N)$$

  is a diffeomorphism

Inverse Function Theorem

Theorem

If $O \subset \mathbb{R}^n$ is open and $F : O \to \mathbb{R}^n$ is a C^k map, where $1 \leq k \leq \infty$ such that at $x_0 \in O$, the linear map

$$\partial F(x_0) : \mathbb{R}^n \to \mathbb{R}^n$$

is invertible, then, if $y_0 = F(x_0)$, there exists $\epsilon > 0$ such that

$$F : F^{-1}(B(y_0, \epsilon)) \to B(y_0, \epsilon)$$

is a diffeomorphism.

- First, we want to show that there exists an $\epsilon > 0$ such that if $y \in B(y_0, \epsilon)$, then there is solution to the equation

$$F(x) = y$$

- Second, we show that the solution is unique
- This defines a map

$$G : B(y_0, \epsilon) \to F^{-1}(B(y_0, \epsilon))$$

Operator Norm of Matrix

- Given a linear map $L : \mathbb{R}^m \to \mathbb{R}^n$, let
$$|L| = \sup |L(x)| \ : \ |x| = 1\}$$
Since L is continuous and the unit sphere is compact, $|L|$ always exists
- It follows that for any $x \in \mathbb{R}^m$
$$\begin{aligned} |L(x)| &= \left| L\left(|x| \frac{x}{|x|} \right) \right| \\ &= \left| L\left(\frac{x}{|x|} \right) \right| |x| \\ &\leq |L||x| \end{aligned}$$
- Properties
$$\begin{aligned} |cL| &= c|L| && \text{(Homogeneity)} \\ |L_1 + L_2| &\leq |L_1| + |L_2| && \text{(Triangle Inequality)} \end{aligned}$$
- A set S of n-by-m matrices is defined to be **open** if for any $L_0 \in S$, there exists $\delta > 0$ such that

Set of Invertible Matrices is Open

- Let $L_0 : \mathbb{R}^m \to \mathbb{R}^m$ be an invertible linear map
- **Lemma:** There exists $\delta > 0$ such that for any linear map $L : \mathbb{R}^m \to \mathbb{R}^m$,
$$|L - L_0| < \delta \implies L \text{ is invertible}$$
- **Proof:** Since
$$|x| = |L_0^{-1}(L_0(x))| \leq |L_0^{-1}||L_0(x)|,$$
it follows that
$$\begin{aligned} L(x) &= L_0(x) + (L - L_0)x \\ &\geq |L_0(x)| - |(L - L_0)x| \\ &\geq |L_0^{-1}|^{-1}|x| - |L - L_0||x| \end{aligned}$$
Therefore, if
$$|L - L_0| < \frac{1}{2}|L_0^{-1}|^{-1},$$
then for any nonzero $x \in \mathbb{R}^m$, $|Lx| > 0$, which implies that L is invertible

Invertible Jacobian (Part 1)

- Let $x_0 \in O$ and assume $\partial F(x_0)$ is invertible
- There exists $\epsilon > 0$ such that for any linear map $M : \mathbb{R}^m \to \mathbb{R}^m$,

$$|M - \partial F(x_0)| < \epsilon \implies M \text{ is invertible}$$

- Since ∂F is continuous, there exists $\delta > 0$ such that

$$x \in B(x_0, \delta) \implies |\partial F(x) - \partial F(x_0)| < \epsilon$$

- For any $x_1, x_2 \in B(x_0, \delta)$ and $t \in [0, 1]$,

$$\begin{aligned} |(x_1 + t(x_2 - x_1)) - x_0| &= |(1 - t)(x_1 - x_0) + t(x_2 - x_0)| \\ &\leq (1 - t)|x_1 - x_0| + t|x_2 - x_0| \\ &\leq \delta, \end{aligned}$$

which implies that $\partial F(x_1 + t(x_2 - x_1))$ is invertible

Invertible Jacobian (Part 1)

- Let
$$M = \int_{t=0}^{t=1} \partial F(x_1 + t(x_2 - x_1))\, dt$$
- Then
$$\begin{aligned} |M - \partial F(x_0)| & \left| \int_{t=0}^{t=1} \partial F(x_1 + t(x_2 - x_1))\, dt - \partial F(x_0) \right| \\ &= \left| \int_{t=0}^{t=1} \partial F(x_1 + t(x_2 - x_1)) - \partial F(x_0)\, dt \right| \\ &\leq \int_{t=0}^{t=1} |\partial F(x_1 + t(x_2 - x_1)) - \partial F(x_0)|\, dt \\ &\leq \int_{t=0}^{t=1} \epsilon\, dt \\ &= \epsilon \end{aligned}$$
- Therefore, M is invertible and
$$|M - \partial F(x_0)| \leq \epsilon$$

Fundamental Theorem of Calculus for a Map

- By the Fundamental Theorem of Calculus and the chain rule, for any $x_1, x_2 \in B(x_0, \delta)$,

$$
\begin{aligned}
F(x_2) - F(x_1) &= \int_{t=0}^{t=1} \frac{d}{dt} F(x_1 + t(x_2 - x_1))\, dt \\
&= \int_{t=0}^{t=1} (x_2^k - x_1^k) \partial_k F(x_1 + t(x_2 - x_1))\, dt \\
&= \left(\int_{t=0}^{t=1} \partial F(x_1 + t(x_2 - x_1))\, dt \right) (x_2 - x_1) \\
&= M(x_2 - x_1)
\end{aligned}
$$

- Therefore,

$$
\begin{aligned}
|F(x_2) - F(x_1)| &\le |M||x_2 - x_1| \\
|x_2 - x_1| &\le |M^{-1}||F(x_2) - F(x_1)|
\end{aligned}
$$

Existence of Solution (Part 1)

- Given any y near $F(x_0)$, we want to show that there exists x near x_0 such that
$$F(x) = y$$
- Recall that $L = \partial F(x_0)$ satisfies
$$F(x) \simeq F(x_0) + L(x - x_0)$$
and therefore, if L is invertible,
$$x \simeq x_0 + L^{-1}(F(x) - F(x_0))$$
- Therefore, an approximate solution is
$$x_1 = x_0 + L^{-1}(y - F(x_0))$$
- More generally, given an approximate solution x_k, we can hope that
$$x_{k+1} = x_k + L^{-1}(y - F(x_k))$$
is a better one
- Observe that $x_{k+1} = x_k$ if and only if x_k is a solution

Reformulation as Fixed Point Equation

- Define a map Φ as follows:

$$\Phi(x) = x + L^{-1}(y - F(x))$$

- Then $F(x) = y$ if and only if

$$\Phi(x) = x$$

Contraction Property

- Observe that

$$\begin{aligned}|\Phi(x_2) - \Phi(x_1)| &= |x_2 - x_1 - L^{-1}(F(x_2) - F(x_1))| \\ &= |x_2 - x_1 - L^{-1}M(x_2 - x_1) \\ &= |L^{-1}(L - M)(x_2 - x_1)| \\ &= |L^{-1}|M - L||x_2 - x_1| \\ &\leq |L^{-1}|\epsilon|x_2 - x_1|\end{aligned}$$

- Choose $\epsilon > 0$ so that $c = |L^{-1}|\epsilon < 1$
- Then

$$|\Phi(x_2) - \Phi(x_1)| \leq c|x_2 - x_1|$$

Contraction Map has Fixed Point (Part 1)

- For each $k \geq 1$, let
$$x_k = \Phi(x_{k-1})$$
- Then
$$\begin{aligned} |x_{k+1} - x_k| &\leq |\Phi(x_k) - \Phi(x_{k-1})| \\ &\leq c|x_k - x_{k-1}| \end{aligned}$$
- By induction,
$$|x_k - x_{k-1}| \leq c^{k-1}|x_k - x_0|$$

Contraction Map has Fixed Point (Part 2)

- Therefore, for any $1 \leq j \leq k$,

$$\begin{aligned}
|x_k - x_j| &= |(x_k - x_{k-1}) + (x_{k-1} - x_{k-2}) + \cdots + (x_{j+1} - x_j)| \\
&\leq |x_k - x_{k-1}| + |x_{k-1} - x_{k-2}| + \cdots + |x_{j+1} - x_j)| \\
&\leq (c^{k-1} + c^{k-1} + \cdots + c^j)|x_1 - x_0| \\
&= c^j \frac{1 - c^j}{1 - c^k} |x_1 - x_0| \\
&\leq c^j |x_1 - x_0|
\end{aligned}$$

- This implies that the sequence $x_0, x_2, \ldots$ is a Cauchy sequence

Differentiability of F^{-1}

- If $x_1 = F^{-1}(y_1)$ and $x_2 = F^{-1}(y_2)$, then

$$F^{-1}(y_2) - F^{-1}(y_1) - L^{-1}(y_2 - y_1) = x_2 - x_1 - L^{-1}(F(x_2) - F(x_1))$$

- Therefore,

$$\begin{aligned}
&\frac{|F^{-1}(y_2) - F^{-1}(y_1) - L^{-1}(y_2 - y_1)|}{|y_2 - y_1|} \\
&= \frac{|x_2 - x_1 - L^{-1}(F(x_2) - F(x_1))|}{|F(x_2) - F(x_1)|} \\
&\leq C|F(x_2) - F(x_1)| \\
&= C|y_2 - y_1|
\end{aligned}$$