MATH-GA1002 Multivariable Analysis

Differentiable Maps
Directional Derivative of Map
Differential of Map
Immersions, Embeddings, Submersions, Diffeomorphisms

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Differentiable Map F from $\mathbb{R}^n$ to $\mathbb{R}^m$

- Given an open $O \subset \mathbb{R}^n$, a map $F : O \to \mathbb{R}^m$ is **differentiable** at $x_0 = (x_0^1, \ldots, x_0^n) \in O$ if there exists a linear map

$$L : \mathbb{R}^n \to \mathbb{R}^m$$

such that

$$\lim_{x \to x_0} \frac{|F(x) - F(x_0) - L(x - x_0)|}{|x - x_0|} = 0$$

- L is called the **differential** or **Jacobian** of F at $x_0 \in O$ and denoted

$$\partial F(x_0) : \mathbb{R}^n \to \mathbb{R}^m$$

- L is also called the **pushforward map** of F at x_0 and denoted

$$F_* : \mathbb{R}^n \to \mathbb{R}^m$$

Directional Derivative of a Map

- The **directional derivative** of $F : O \to \mathbb{R}^m$ at $x_0 \in O$ in the direction $v \in \mathbb{R}^n$ is defined to be $D_v F(x_0) \in \mathbb{R}^m$, where

$$\begin{aligned} D_v F(x_0) &= \left.\frac{d}{dt}\right|_{t=0} F(x_0 + tv) \\ &= \lim_{t \to 0} \frac{F(x_0 + tv) - f(x_0)}{t} \end{aligned}$$

- If F is differentiable at x_0 and its differential at x_0 is L, then if $v \neq 0$, it follows by the definition of the differential of F that

$$\begin{aligned} 0 &= \lim_{t \to 0} \frac{|F(x_0 + tv) - f(x_0) - L(tv)|}{|tv|} \\ &= \lim_{t \to 0} \frac{1}{|v|} \left| \frac{F(x_0 + tv) - F(x_0)}{t} - L(v) \right| \\ &= \frac{1}{|v|} |D_v f(x_0) - L(v)| \end{aligned}$$

Differential of a Map

- Therefore, the directional derivative of F at x_0 in the direction v is given by

$$D_v F(x_0) = L(v) = \partial F(x_0)(v)$$

- Equivalently, the differential of F at x_0 is the linear map

$$\begin{aligned} \partial F(x_0) : \mathbb{R}^n &\to \mathbb{R}^m \\ v &\mapsto D_v F(x_0) \end{aligned}$$

- Equivalently, the pushforward map of F at x_0 is the linear map

$$\begin{aligned} F_* : \mathbb{R}^n &\to \mathbb{R}^m \\ v &\mapsto D_v F(x_0) \end{aligned}$$

Smooth Maps

- A map $F = (f^1, \ldots, f^m) : O \to \mathbb{R}^m$ is C^k if each f_j is C^k, $1 \leq j \leq m$
- Lemma: *If F is C^1, then it is differentiable*
- The space of all C^k maps with domain O and range $\mathbb{R}^m$ is denoted $C^k(O, \mathbb{R}^m)$

Jacobian of a C^1 Map

- Let $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ denote the space of a linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$
- Let $F : O \to \mathbb{R}^m$ be a C^1 map and $\partial F : O \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ its Jacobian
- For each $x \in O$, the directional derivative of F at x in the direction v is defined to be

$$D_v F(x) = \left.\frac{d}{dt}\right|_{t=0} F(c(t)) \in \mathbb{R}^m,$$

  where c is a C^1 curve such that $c(0) = x$ and $c'(0) = v$
- It follows by the same proof as for C^1 functions that

$$D_v F(x) = (\partial F(x))(v)$$

- It is also called the **pushforward** of F, denoted

$$\begin{aligned} F_* : \mathbb{R}^n &\to \mathbb{R}^m \\ v &\mapsto D_v F(x) \end{aligned}$$

Example: Polar Coordinates

- Polar coordinates: Let

$$
\begin{aligned}
P : \mathbb{R}^2 &\to \mathbb{R}^2 \\
(r,\theta) &\mapsto (r\cos(\theta), r(\sin\theta))
\end{aligned}
$$

- Given $(v, w) \in \mathbb{R}^2$, the directional derivative of P at (r_0, θ_0) is

$$
\begin{aligned}
&D_{(v,w)}P(r_0,\theta_0) \\
&= \frac{d}{dt}\bigg|_{t=0} ((r_0+tv)\cos(\theta_0+tw), (r_0+tv)\sin(\theta_0+tw)) \\
&= (v\cos(\theta_0+tw) - (r_0+tv)w\sin(\theta_0+tw), \\
&\qquad v\sin(\theta_0+tw) + ((r_0+tv)w\cos(\theta_0+tw))\big|_{t=0} \\
&= (v\cos(\theta_0) - wr_0\sin(\theta_0), v\sin(\theta_0) + wr_0\cos(\theta_0)) \\
&= v(\cos(\theta_0), \sin(\theta_0)) + w(-r_0\sin(\theta_0), r_0\cos(\theta_0))
\end{aligned}
$$

Pushforward of Polar Coordinate Map

- The Jacobian of P is given by

$$\begin{aligned}(\partial P(r_0,\theta_0))(v,w) &= D_{(v,w)}P(r_0,\theta_0)\\ &= \begin{bmatrix}\cos(\theta_0) & -r_0\sin(\theta_0)\\ \sin(\theta_0) & r_0\cos(\theta_0)\end{bmatrix}\begin{bmatrix}v\\ w\end{bmatrix}\end{aligned}$$

- The pushforward map of P at (r_0,θ_0) is

$$\begin{aligned}P_*(v,w) &= D_{(v,w)}P(r_0,\theta_0)\\ &= \begin{bmatrix}\cos(\theta_0) & -r_0\sin(\theta_0)\\ \sin(\theta_0) & r_0\cos(\theta_0)\end{bmatrix}\begin{bmatrix}v\\ w\end{bmatrix}\end{aligned}$$

- Treat and calculate the Jacobian or pushforward as a **map** and not a matrix

Differentiation of a Matrix Function of a Single Variable

- Let $\mathrm{gl}(n)$ denote the space of real n-by-n matrices and $\mathrm{GL}(n)$ the space of invertible matrices
- Let I be a nonempty connected interval and consider a map

$$F : I \to \mathrm{gl}(n) = \mathbb{R}^{n^2}$$

- Let $F^j_k : I \to \mathbb{R}$ denote the component in the j-th row and k-th column of F
- F is defined to be C^k if each F^j_k is C^k
- If F is C^1, then the derivative of F is the map $F' : I \to \mathrm{gl}(n)$, where the component of F' in the j-th row and k-th column is

$$(F')^k_k = (F^j_k)'$$

Rules of Differentiation

- Let $c^1, c^2 \in \mathbb{R}$ and F_1, F_2 be C^1 maps from I to $\mathrm{gl}(n)$
- Constant factor and sum rules: Since, for each $1 \le j, k \le n$,

$$(c^1(F_1)^j_k + c^2(F_2)^j_k)' = c^1(F_1')^j_k + c^2(F_2')^j_k,$$

  it follows that

$$(c^1 F_1 + c^2 F_2)' = c^1 F_1' + c^2 F_2'$$

- Product rule: Since, for each $1 \le j, k, l \le n$,

$$((F_1)^j_k (F_2)^k_l)' = (F_1')^j_k (F_2)^k_l + (F_1)^j_k (F_2')^k_l,$$

  it follows by summing this over $1 \le k \le n$ and the sum rule that

$$\left(\sum_{k=1}^{n} ((F_1)^j_k (F_2)^k_l\right)' = \sum_{k=1}^{n} ((F_1')^j_k (F_2)^k_l + (F_1)^j_k (F_2')^k_l)$$

- Therefore,

$$(F_1 F_2)' = F_1' F_2 + F_1 F_2'$$

Implicit Differentiation of Matrix Inverse (Part 1)

- Recall that $\mathrm{GL}(n)$ is an open subset of $\mathrm{gl}(n) = \mathbb{R}^{n^2}$
- We want to find the differential of the map

$$\begin{aligned} \Phi : \mathrm{GL}(n) &\to \mathrm{GL}(n) \\ M &\mapsto M^{-1} \end{aligned}$$

- The map

$$\begin{aligned} \Psi : \mathrm{GL}(n) &\to \mathrm{GL}(n) \\ M &\mapsto M\Phi(M) \end{aligned}$$

  is the constant map because $\Psi(M) = I$ for all $M \in \mathrm{GL}(n)$
- Therefore, the differential of Ψ is always the zero map

Implicit Differentiation of Matrix Inverse (Part 2)

- For any $M \in \mathrm{GL}(n)$ and $V \in \mathrm{gl}(n)$,

$$
\begin{aligned}
0 &= (\partial\Psi(M))(V) \\
&= D_V\Psi(M) \\
&= \left.\frac{d}{dt}\right|_{t=0} \Psi(M+tV) \\
&= \left.\frac{d}{dt}\right|_{t=0} ((M+tV)\Phi(M+tV)) \\
&= \left(\left.\frac{d}{dt}\right|_{t=0} ((M+tV) \right) \Phi(M)) + M \left(\left.\frac{d}{dt}\Phi(M+tv)\right|_{t=0} \right) \\
&= V\Phi(M) + MD_V\Phi(M) \\
&= VM^{-1} + MD_V\Phi(M)
\end{aligned}
$$

- Therefore, at $M \in \mathrm{GL}(n)$, the pushforward maps is

$$
\begin{aligned}
\Phi_* : \mathrm{gl}(n) &\to \mathrm{gl}(n) \\
V &\mapsto -M^{-1}VM^{-1}
\end{aligned}
$$

Immersions, Embeddings, Diffeomorphisms

- Let $O \subset \mathbb{R}^n$ be open and $F : O \to \mathbb{R}^m$ be C^1
- If, for any $x \in O$, the Jacobian of F at x,
$$\partial F(x) : \mathbb{R}^n \to \mathbb{R}^m,$$
is injective, then F is an **immersion**
- If F is an injective immersion, it is called an **embedding**
- If $m = n$ and F is an embedding, then the map
$$F : O \to F(O)$$
is bijective and called a **diffeomorphism**
- Fact: If for some $1 \leq k \leq \infty$, $F : O \to F(O)$ is a C^k diffeomorphism, then its inverse map
$$F^{-1} : F(O) \to O$$
is also a C^k diffeomorphism

Linear Isomorphism Implies Local Nonlinear Isomorphism

- Let V and W be n-dimensional vector spaces
- If $L : V \to W$ is a linear map with maximal rank, then
  - L is bijective
  - Its inverse map $L^{-1} : W \to V$ is linear
  - I.e., L and L^{-1} are linear isomorphisms
- Let $O \subset \mathbb{R}^n$ be open and $\Phi : O \to \mathbb{R}^n$ be a C^1 map
- Suppose that at $x_0 \in O$, the differential of Φ

$$\partial\Phi(x_0) : \mathbb{R}^n \to \mathbb{R}^n$$

  is a linear isomorphism
- Then there exists an open neighborhood $N \subset O$ of x_0 such that

$$\Phi|_N : N \to \Phi(N)$$

  is a diffeomorphism