MATH-GA1002 Multivariable Analysis

Topology of $\mathbb{R}^n$
Continuous Functions and Maps
Differentiable Functions and Maps
Differential of a Function
Directonal Derivative of a Function

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Topology of $\mathbb{R}^n$: Open Sets

- For any $x_1, x_2 \in \mathbb{R}^n$,

$$|x_2 - x_1| = \left(\sum_{k=1}^{n}(x_2 - x_1)^2\right)^{1/2}$$

$$|x_2 - x_1|_\infty = \max(|x_2^1 - x_1^1|, \ldots, |x_2^n - x_1^n|)$$

- An **open ball** of radius r centered at $x_0 \in \mathbb{R}^n$ is defined to be

$$B(x_0, r) = \{x \in \mathbb{R}^n \; : \; |x - x_0| < r\}$$

- An **open cube** of radius r centered at $x_0 \in \mathbb{R}^n$ is defined to be

$$C(x_0, r) = \{x \in \mathbb{R}^n \; : \; |x - x_0|_\infty < r\}$$

- A set $O \subset \mathbb{R}^n$ is **open** if for any $x \in O$, there exists $r > 0$ such that $C(x, r) \subset O$
- Equivalently, a set $O \subset \mathbb{R}^n$ is **open** if for any $x \in O$, there exists $r > 0$ such that $B(x, r) \subset O$

Closed Sets

- A set $C \subset \mathbb{R}^n$ is **closed** if $\mathbb{R}^n \backslash C$ is open
- Examples:
  - The **closed cube**

$$\overline{C(x_0, r)} = \{x \in \mathbb{R}^n \ : \ |x - x_0|_\infty \leq r\}$$

  - The **closed ball**

$$\overline{B(x_0, r)} = \{x \in \mathbb{R}^n \ : \ |x - x_0| \leq r\}$$

- A set $S \subset \mathbb{R}^n$ that is not open is **not** necessarily closed

Bounded Sets, Compact Sets

- A set $S \subset \mathbb{R}^n$ is **bounded** if there exists a cube $C(0, R)$ such that $S \subset C(0, R)$
- A set $S \subset \mathbb{R}^n$ is **compact** if any sequence in S has a convergent subsequence
- (Heine-Borel) A set $S \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded
- Basic examples: Closed balls and closed cubes
- Lemma: $S \subset \mathbb{R}^n$ is compact if and only if any open covering of S has a finite subcover
  - An open covering of S is a possibly infinite collection of open sets whose union contains S
  - A finite subcover is a finite number of open sets in the collection whose union contains S

Bounded and Continuous Maps

- A map $f : S \to \mathbb{R}^m$ is **bounded** if there exists $M > 0$ such that
$$\forall s \in S, \ |f(s)| \leq M$$
- Given $S \subset \mathbb{R}^n$, a function $f : S \to \mathbb{R}^m$ is **continuous** at $x_0 \in S$ if the following holds:
  - For any $\epsilon > 0$, there exists $\delta > 0$ such that
$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$
- $f : S \to \mathbb{R}^m$ is continuous if it is continuous at each $x_0 \in S$
- Lemma: $f : S \to \mathbb{R}^m$ is continuous if and only if for any open set $O' \subset \mathbb{R}^m$, there exists an open $O \subset \mathbb{R}^n$ such that
$$f^{-1}(O') = O \cap S$$

Differentiable Function on $\mathbb{R}$

- Rough idea: A function is differentiable at a point if it has a good linear approximation at that point
- Given an open interval $I \subset \mathbb{R}$, a function $f : I \to \mathbb{R}$ is **differentiable** at $x_0 \in I$ if there exists $f'(x_0) \in \mathbb{R}$ such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

  $f'(x_0)$ is called the **derivative** of f at x_0
- Equivalently, f is differentiable at $x_0 \in I$ if there exists $m \in \mathbb{R}$ such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - m(x - x_0)}{x - x_0} = 0$$

  If so, the derivative of f at x_0 is defined to be

$$f'(x_0) = m$$

Differentiable Function on $\mathbb{R}^n$

- Rough idea: A function is differentiable at a point if it has a good linear approximation at that point
- Given an open $O \subset \mathbb{R}^n$, a function $f : O \to \mathbb{R}$ is **differentiable** at $x_0 = (x_0^1, \ldots, x_0^n) \in O$ if there exists a linear function $\ell : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{x \to x_0} \frac{|f(x) - f(x_0) - \ell(x - x_0)|}{|x - x_0|} = 0$$

- Observe that $\ell \in (\mathbb{R}^n)^*$
- ℓ is called the **differential** of f at $x_0 \in O$ and denoted $df(x_0)$
- The differential of f is a map

$$df : O \to (\mathbb{R}^n)^*$$

Directional Derivative of a Function

- The **directional derivative** of $f : O \to \mathbb{R}$ at $x_0 \in O$ in the direction $v \in \mathbb{R}^n$ is defined to be

$$\begin{aligned} D_v f(x_0) &= \left.\frac{d}{dt}\right|_{t=0} f(x_0 + tv) \\ &= \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} \end{aligned}$$

- If f is differentiable at x_0 and its differential at x_0 is $\ell = df(x_0)$, then if $v \neq 0$,

$$\begin{aligned} 0 &= \lim_{t \to 0} \frac{|f(x_0 + tv) - f(x_0) - \ell(tv)|}{|tv|} \\ &= \lim_{t \to 0} \frac{1}{|v|} \left| \frac{f(x_0 + tv) - f(x_0)}{t} - \ell(v) \right| \\ &= \frac{1}{|v|} |D_v f(x_0) - \ell(v)| \end{aligned}$$

- Therefore, $D_v f(x_0) = \langle df(x_0), v \rangle$

Partial Derivatives

- Let $(e_1, \ldots, e_n)$ denote the standard basis of $\mathbb{R}^n$
- If $f : O \to \mathbb{R}$ is differentiable at $x_0 \in O$, then the k-th partial derivative of f at x_0 is defined to be

$$\partial_i f(x_0) = \left.\frac{d}{dt}\right|_{t=0} f(x_0 + te_k) = \lim_{t \to 0} \frac{f(x_0 + te_i) - f(x_0)}{t}$$

- If ℓ is the differential of f at x_0, then

$$\partial_k f(x_0) = \ell(e_k) = \langle df(x_0), e_k \rangle$$

- Conversely, if $v = e_k v^k$, then

$$\begin{aligned} \langle df(x_0), v \rangle &= \langle df(x_0), e_k v^k \rangle \\ &= v^k \langle df(x_0), e_k \rangle \\ &= v^k \partial_k f(x_0) \end{aligned}$$

Chain Rule

- Let $O \subset \mathbb{R}^n$ be open and $f : O \to \mathbb{R}$ be differentiable at $x_0 \in O$
- Let $I \subset \mathbb{R}$ be a connected open interval and $c : I \to O$ be a curve such that at $t_0 \in I$,

$$c(t_0) = x_0 \text{ and } c'(t_0) = v$$

- The derivative of the composition $f \circ c : I \to \mathbb{R}$ at t_0 is

$$\begin{aligned}
(f \circ c)'(t_0) &= \left.\frac{d}{dt}\right|_{t=t_0} f(c^1(t), \ldots, c^n(t)) \\
&= \sum_{k=1}^{n} (\partial_k f(c(t_0)))(c^k)'(t_0) \\
&= v^k \partial_k f(x_0) \\
&= \langle df(x_0), v \rangle \\
&= D_v f(x_0)
\end{aligned}$$

Differential of Coordinate Function

- The differential of x^i at each $x_0 \in \mathbb{R}^n$ satisfies for any $v = (v^1, \dots, v^n) \in \mathbb{R}^n$

$$\begin{aligned}\langle dx^i(x_0), v\rangle &= \left.\frac{d}{dt}\right|_{t=0} x^i(x_0 + tv) \\ &= \left.\frac{d}{dt}\right|_{t=0} (x^i + tv^i) \\ &= v^i\end{aligned}$$

- In particular, if $(\epsilon^1, \dots, \epsilon^n)$ is the dual basis to the standard basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$, then

$$dx^i = \epsilon^i$$