MATH-GA1002 Multivariable Analysis

Linear and Multilinear Algebra
1-Tensors, 2-tensors

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Course Information

- Web Pages
  - My homepage: https://math.nyu.edu/∼yangd
  - Course Hompage
  - Course Calendar
- Textbook
  - Michael Spivak, **Calculus on Manifolds**
  - Available as resource on Ed Resources

Definition of Real Vector Space

- A set V with the following binary operations
  - **Vector addition:**

$$\begin{aligned} V \times V &\to V \\ (v_1, v_2) &\mapsto v_1 + v_2 \end{aligned}$$

  - **Scalar multiplication**

$$\begin{aligned} \mathbb{R} \times V &\to V \\ (s, v) &\mapsto sv = vs \end{aligned}$$

Required Properties of Vector Addition

- Associativity of vector addition: For any $v_1, v_2, v_3 \in V$,

$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$$

- Commutativity: For any $v_1, v_2 \in V$,

$$v_1 + v_2 = v_2 + v_1$$

- Additive identity: There exists an element, denoted $0 \in V$, such that for any $v \in V$,

$$v + 0 = v$$

- Additive inverse: For any $v \in V$, there exists an element, denoted $-v \in V$, such that

$$v + (-v) = 0$$

Required Properties of Scalar Multiplication

- Multiplicative identity: For any $v \in V$,

$$1v = v1 = v$$

- Associativity of scalar multiplication: For any $s_1, s_2 \in \mathbb{R}$ and $v \in V$,

$$(s_1 s_2)v = s_1(s_2 v)$$

- Distributivity: For any $s \in \mathbb{R}$, $v_1, v_2 \in V$,

$$s(v_1 + v_2) = sv_1 + sv_2$$

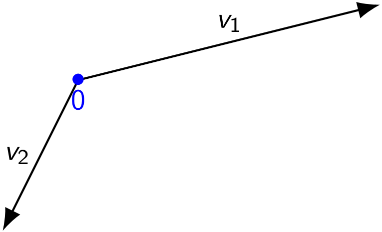
- Distributivity: For any $s_1, s_2 \in \mathbb{R}$, $v \in V$,

$$(s_1 + s_2)v = s_1 v + s_2 v$$

Vectors in the Plane

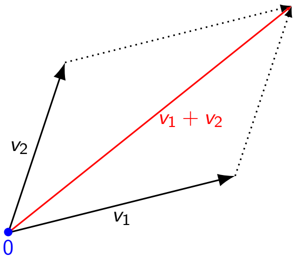
- There is a zero vector, also called the origin

- A vector is an arrow that starts at the origin and has a direction and magnitude (length)

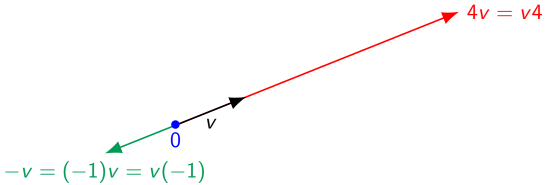


Vector Operations

- Vector addition



- Scalar multiplication



Span, Linear Independence and Basis

- The **span** of a subset $S \subset V$ is

  $$[S] = \{s^1 v_1 + \cdots + s^k v_k \ : \ \forall s_1, \ldots, s_k \in \mathbb{R}, \ \forall v_1, \ldots, v_k \in S, \ \forall k \in \mathbb{Z}_+$$

- A finite set $\{v_1, \ldots, v_n\} \subset V$ is **linearly independent** if for any $s^1, \ldots, s^n \in \mathbb{R}$,

  $$s^1 v_1 + \cdots + s^n v_n = 0 \implies s^1 = \cdots = s^n = 0.$$

- A **basis** of V is a linearly independent set

  $$B = \{b_1, \ldots, b_n\} \subset V$$

  such that $[B] = V$
- A vector space is finite-dimensional if it has a basis, and its dimension is defined to be the number of elements in the basis
  - Any two bases have the same number of elements

Subspace

- $S \subset V$ is a **linear subspace** if, using the same vector addition and scalar multiplication as V, it is itself a vector space
- I.e., for any $s^1, s^2 \in \mathbb{R}$ and $v_1, v_2 \in S$,

$$s^1 v_1 + s^2 v_2 \in S$$

- A linear subspace can be viewed as a plane in V that passes through 0

Quotient Space

- Given a subspace $S \subset V$, an **affine subspace** parallel to S is a plane parallel to S that does not necessarily pass through 0
- An affine subspace parallel to S that contains $v \in V$ is the set

$$v + S = \{v + w \ : \ w \in S\}$$

- That $v + S$ is parallel to S means

$$v_1 + S = v_2 + S \iff v_2 - v_1 \in S$$

- The **quotient space** of V over S is defined to be the set of all affine subspaces parallel to S,

$$V/S = \{v + S \ : \ v \in V\}$$

- V/S is a vector space, where for any $v_1, v_2, v \in V$ and $t \in \mathbb{R}$,

$$(v_1 + S) + (v_2 + S) = (v_1 + v_2) + S$$
$$t(v + S) = tv + S$$

Linear Maps

- If V and W are vector spaces, then a map $L : V \to W$ is **linear** if for any $s^1, s^2 \in \mathbb{R}$ and $v_1, v_2 \in V$,

$$L(s^1 v_1 + s^2 v_2) = s^1 L(v_1) + s^2 L(v_2)$$

Injective, Surjective, Bijective Linear Maps

- Consider a linear map $L : V \to W$
- L is **injective** if

$$L(v_1) = L(v_2) \implies v_1 = v_2$$

  - $L : V \to W$ is injective if and only if $\ker L = \{0\}$
- $L : V \to W$ is **surjective** if $L(V) = W$
- A map $L : V \to W$ is **bijective** or an **isomorphism** if it is both injective and surjective
- If $L : V \to W$ is a linear isomorphism, then so is $L^{-1} : W \to W$
- A basis $(v_1, \ldots, v_n)$ of V defines a linear isomorphism

$$\mathbb{R}^n \to V$$
$$(s^1, \ldots, s^n) \mapsto s^1 v_1 + \cdots + s^n v_n$$

Kernel, Image, Cokernel of Linear Map

- The **kernel** of L is

$$\ker(L) = \{v \in V \ : \ L(v) = 0\} \subset V$$

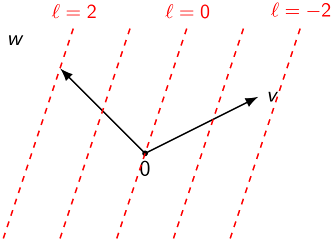
- The **image** of L is

$$L(V) = \{w \in W \ : \ \exists v \in V \text{ such that } L(v)w\} = \{L(v) \ : \ \forall v \in V\}$$

- The **cokernel** is the quotient space $V/(\ker L)$
- There is a natural linear isomorphism

$$\begin{aligned} V/(\ker L) &\to L(V) \\ v + \ker(L) &\mapsto L(v) \end{aligned}$$

Dual Vector Space



- The dual vector space of a finite dimensional vector space V is the vector space of linear functions on V,

$$V^* = \{\ell : V \to \mathbb{R} \ : \ \ell \text{ is linear}\}$$

- An element of V^* is called a **covector** or a 1-**tensor**

Dual Basis

- Let $(v_1, \dots, v_n)$ be a basis of V
- Any $\ell \in V^*$ is uniquely determined by its values $\ell(v_1), \dots, \ell(v_n)$, because

$$\ell(s^1 v_1 + \cdots + s^n v_n) = s^1 \ell(v_1) + \cdots + s^n \ell(v_n)$$

- The basis $(v_1, \dots, v_n)$ naturally induces a basis $(\ell^1, \dots, \ell^n)$ of V^* such that

$$\ell^j(v_k) = \delta^j_k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

Pushforward of a Vector and Pullback of a Covector

- Let $F : V \to W$ be a linear map
- Given a vector $v \in V$, the **pushforward** of v is $F(v) \in W$
- Given a covector $\ell \in W$, the **pullback** of $\ell \in W^*$ is the element $F^*\ell \in V^*$, where

$$F^*\ell = \ell \circ F : V \to \mathbb{R}$$

- The pullback of F is a linear map

$$\begin{aligned} F^* : W^* &\to V^* \\ \ell &\mapsto \ell \circ F \end{aligned}$$

Dual of Dual of a Vector Space is Itself

- There is a natural linear isomorphism $V^{**} \to V$
- Any $v \in V$ defines a linear function

$$\begin{aligned} f_v : V^* &\to \mathbb{R} \\ \ell &\mapsto \ell(v) \end{aligned}$$

- This defines a map

$$\begin{aligned} F : V &\to V^{**} \\ v &\mapsto f_v \end{aligned}$$

- Using a basis of V and its dual basis, it is easy to show that F is a linear isomorphism

Notation

- If $v \in V$ and $\ell \in V^*$, we will write

$$\langle \ell, v \rangle = \ell(v) = v(\ell) = \langle v, \ell \rangle$$

- $v \in V$ is called a **vector**
- $\ell \in V^*$ is called a **covector** or a **1-tensor**

2-tensors

- A function $f : V \times V \to \mathbb{R}$ is **bilinear** if for any $w \in V$, the functions

$$\begin{aligned} V &\to \mathbb{R} \\ v &\mapsto f(v, w) \end{aligned}$$

  and

$$\begin{aligned} V &\to \mathbb{R} \\ v &\mapsto f(w, v) \end{aligned}$$

  are linear
- Equivalently, $f : V \times W$ is bilinear if for any $a^1, a^2, b^1, b^2 \in R$ and $v, v_1, v_2, w, w_1, w_2 \in V$,

$$\begin{aligned} f(a^1 v_1 + a^2 v_2, w) &= a^1 f(v_1, w) + a^2 f(v_2, w) \\ f(v, b^1 w_1 + b^2 w_2) &= b^1 f(v, w_1) + b^2 f(v, w_2) \end{aligned}$$

- A bilinear function is also called a 2-**tensor**

Vector Space of 2-Tensors

- The space of 2-tensors on V is denoted

$$V^* \otimes V^* = \{ \text{ bilinear functions } f : V \times V \to \mathbb{R}\},$$

  which is a vector space
- If $f_1, f_2 \in V^* \otimes V^*$ and $a^1, a^2 \in \mathbb{R}$, then

$$a^1 f_1 + a^2 f_2 \in V^* \otimes V^*$$

- Therefore, $V^* \otimes V^*$ is a vector space
- There is a natural map

$$\begin{gathered} V * \times V^* \to V^* \otimes V^* \\ (\ell^1, \ell^2) \mapsto \ell^1 \otimes \ell^2, \end{gathered}$$

  where for any $v_1, v_2 \in V$,

$$(\ell^1 \otimes \ell^2)(v_1, v_2) = \langle \ell^1, v_1 \rangle \langle \ell^2, v_2 \rangle$$

Pullback of a 2-Tensor

- Given any linear map $L : V \to W$, there is a natural pullback map
$$L^* : W^* \otimes W^* \to V^* \otimes V^*,$$
where for any bilinear function
$$f : W \times W \to \mathbb{R},$$
the bilinear function $L^* f : V \times V \to \mathbb{R}$ is defined to be
$$(L^* f)(v_1, v_2) = f(L(v_1), L(v_2)), \ \forall v_1, v_2 \in V$$

k-Tensors

- A k-tensor is a **multilinear function**

$$f : V \times \cdots \times V \to \mathbb{R},$$

  where, if all inputs but one are held fixed, then f is a linear function of the remaining input
- The space of all k-tensors, denoted $V^* \otimes \cdots \otimes V^*$, is a finite-dimension real vector space
- The space of 1-tensors is V^*