

Orthogonal  
Subspaces and  
Projections

Construction of  
Unitary Basis

Adjoint Maps and  
Matrices

# MA-GY 7043: Linear Algebra II

Orthogonal Subspaces and Projections

Construction of Unitary Basis

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March 2 2026

# Outline I

Orthogonal  
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## Orthogonal Subspaces and Projections

Construction of  
Unitary Basis

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## Adjoint Maps and Matrices

# Direct Sum of Two Subspaces

- ▶ Let  $S, T \subset V$  be subspaces
- ▶  $S, T$  are **transversal** if  $S \cap T = \{0\}$
- ▶ If  $S, T$  are transversal, then if  $s \in S$  and  $t \in T$  satisfy

$$s + t = 0,$$

it follows that  $t = -s \in S$  and therefore  $s, t \in S \cap T = \{0\}$

- ▶ This proves that  $S, T$  transversal implies  $S, T$  are linearly independent
- ▶ Conversely, if  $S$  and  $T$  are linearly independent, then for each  $v \in S \cap T$ , the vectors  $s = v \in S$  and  $t = -v \in T$  satisfy

$$s + t = 0,$$

which implies that  $s = t = 0$ , i.e.,  $v = 0$

- ▶ This proves  $S, T$  linearly independent implies  $S, T$  transversal

# Projection Maps

- ▶ If  $S$  and  $T$  are transversal, then for any  $v \in S \oplus T$ , there are unique vectors  $s \in S$  and  $t \in T$  such that

$$v = s + t$$

- ▶ We can therefore define maps

$$P_S : S \oplus T \rightarrow S$$

$$P_T : S \oplus T \rightarrow T,$$

where for any  $v \in S \oplus T$ ,

$$v = P_S(v) + P_T(v)$$

- ▶ These maps satisfy the following:

$$P_S + P_T = I$$

$$P_S \circ P_S = P_S$$

$$P_T \circ P_T = P_T$$

$$P_S \circ P_T = P_T \circ P_S = 0$$

# Orthogonal Complement

- ▶ Let  $V$  be an inner product space
- ▶ Given a subspace  $E \subset V$ , define its **orthogonal complement** to be the subspace

$$E^\perp = \{v \in V : \forall e \in E, (v, e) = 0\}$$

- ▶  $E, E^\perp$  are transversal because if

$$v \in E \cap E^\perp,$$

then

$$|v|^2 = (v, v) = 0,$$

- ▶ The projection maps

$$P_E : E \oplus E^\perp \rightarrow E$$
$$P_E^\perp = P_{E^\perp} : E \oplus E^\perp \rightarrow E^\perp$$

are called **orthogonal projections**

# Orthogonal Projection Minimizes Distance to a Subspace

- **Fact:** For each  $v \in E \oplus E^\perp$  and  $w \in E$ , the distance from  $v$  to  $w$  is minimized if  $w = P_E(v)$ , i.e.,

$$|v - w| \geq |v - P_E(v)|$$

and equality holds if and only if  $w = P_E(v)$

- **Proof:** Let  $v = v_1 + v_2$ , where

$$v_1 = P_E(v) \in E \text{ and } v_2 = v - P_E(v) \in E^\perp$$

- Then for any  $w \in E$ ,

$$\begin{aligned} |v - w|^2 &= |v_1 + v_2 - w|^2 \\ &= (v_1 - w + v_2, v_1 - w + v_2) \\ &= (v_1 - w, v_1 - w) + (v_1 - w, v_2) + (v_2, v_1 - w) + (v_2, v_2) \\ &= |v_1 - w|^2 + |v_2|^2 \\ &\geq |v_1 - w|^2 + |v - v_1|^2 \geq |v - v_1|^2 \end{aligned}$$

and equality holds if and only if  $w = v_1$

# Orthogonal Projection Using an Orthonormal Set (Part 1)

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- ▶ Assume that the subspace  $E \subset V$  has an orthonormal basis  $(u_1, \dots, u_k)$
- ▶ For any  $v \in E$ , there exist  $a^1, \dots, a^k \in \mathbb{F}$  such that

$$v = a^1 u_1 + \dots + a^k u_k$$

- ▶ Since, for each  $j \in \{1, \dots, k\}$ ,

$$(v, u_j) = (a^1 u_1 + \dots + a^k u_k, u_j) = a^j,$$

it follows that

$$v = (v, u_1)u_1 + \dots + (v, u_k)u_k$$

# Orthogonal Projection Using an Orthonormal Set (Part 2)

- ▶ Consider the map  $P_E : V \rightarrow E$  given by

$$P_E(v) = (v, u_1)u_1 + \cdots + (v, u_k)u_k$$

- ▶ For any  $v \in V$  and  $j \in \{1, \dots, k\}$ ,

$$(v - P_E(v), u_j) = (v, u_j) - (v, u_j) = 0$$

and therefore

$$v - P_E(v) \in E^\perp$$

- ▶ Therefore,  $P_E$  and the map  $P_E^\perp : V \rightarrow E^\perp$  given by

$$P_E^\perp(v) = v - P_E(v),$$

are maps from  $V$  to  $V$  that satisfy

$$P_E + P_E^\perp = I$$

- ▶ They are also projection maps
- ▶ It follows that, if  $E$  has an orthonormal basis, then

$$E \oplus E^\perp = V$$

# Any Inner Product Space has an Orthonormal Basis

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- ▶ Proof by mathematical induction
- ▶ A 1-dimensional inner product space has an orthonormal basis
- ▶ Let  $V$  be an  $n$ -dimensional inner product space
- ▶ Let  $E_k$  be a  $k$ -dimensional subspace of  $V$ , with  $1 \leq k < n$  have an orthonormal basis  $(u_1, \dots, u_k)$
- ▶ Since  $E_k \subsetneq V$ , there exists  $w \in V \setminus E_k$
- ▶ Let  $E_{k+1}$  be the span of  $E_k$  and  $w$
- ▶ Let  $v = P_{E_k}^\perp(w)$  and observe that it is nonzero
- ▶ Set

$$u_{k+1} = \frac{v}{|v|}$$

- ▶ The set  $\{u_1, \dots, u_{k+1}\}$  is an orthonormal basis of  $E_{k+1}$
- ▶ By induction,  $V = E_n$  has an orthonormal basis

# Gram-Schmidt Algorithm

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- ▶ The Gram-Schmidt algorithm is a way to construct an orthonormal basis from a given basis of an inner product space
- ▶ Let  $(w_1, \dots, w_n)$  be a basis of  $V$  and, for each  $1 \leq k \leq n$ , let

$$E_k = \text{span of } (w_1, \dots, w_k)$$

- ▶ Assume there is an orthonormal basis  $(u_1, \dots, u_k)$  of  $E_k$  and let

$$\begin{aligned}v_{k+1} &= P_{E_k}^\perp(w_{k+1}) \\ &= w_{k+1} - P_{E_k}(w_{k+1}) \\ &= w_{k+1} - ((w_{k+1}, u_1)u_1 + \dots + (w_{k+1}, u_k)u_k)\end{aligned}$$

- ▶ Since  $v_{k+1} \neq 0$ , we can set

$$u_{k+1} = \frac{v_{k+1}}{|v_{k+1}|}$$

- ▶  $(u_1, \dots, u_{k+1})$  is an orthonormal basis of  $E_{k+1}$
- ▶ By induction, we obtain an orthonormal basis  $(u_1, \dots, u_n)$  such that for each  $k \in \{1, \dots, n\}$ ,

$$E_k = \text{span of } \{u_1, \dots, u_k\}$$

# Adjoint of a Linear Maps

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- ▶ Let  $V, W$  be inner product spaces and  $L : V \rightarrow W$  be a linear map
- ▶ The **(Hermitian) adjoint** of  $L$  is defined to be the map  $L^* : W \rightarrow V$  such that for any  $v \in V$  and  $w \in W$ ,

$$(L(v), w) = (v, L^*(w))$$

## Adjoint of Linear Map (Part 2)

- ▶ Let

$$E = [e_1 \quad \dots \quad e_n]$$

be a unitary basis of  $V$  and

$$F = [f_1 \quad \dots \quad f_m]$$

be a unitary basis of  $W$

- ▶ A linear map  $L : V \rightarrow W$  is uniquely determined by the matrix  $M$ , where

$$L(E) = FM,$$

- ▶ We want to show that there exists a matrix  $N$  such that the map  $L^* : W \rightarrow V$  given by

$$L^*F = EN$$

is the adjoint of  $L$

# Adjoint of Linear Maps and Matrices (Part 3)

- ▶ For any vectors

$$v = e_1 a^1 + \cdots + e_n a^n = Ea \text{ and } w = f_1 b^1 + \cdots + f_m b^m = Fb,$$

we get

$$\begin{aligned}(L(v), w) &= (LEa, Fb) \\ &= (FMa, Fb) \\ &= (f_p M_j^p a^j, f_q \bar{b}^q) \\ &= (f_p, f_q) M_j^p a^j \bar{b}^q \\ &= \delta_{pq} M_j^p a^j \bar{b}^q \\ &= \sum_{j=1}^m \sum_{p=1}^n M_j^p a^j \bar{b}^p\end{aligned}$$

# Adjoint of Linear Maps and Matrices (Part 4)

- ▶ On the other hand,

$$\begin{aligned}(v, L^*(w)) &= (Ea, L^*(Fb)) \\ &= (Ea, ENb) \\ &= (e_j a^j, e_k N_p^k b^p) \\ &= (e_j, e_k) a^j \bar{N}_p^k \bar{b}^p \\ &= \delta_{jk} a^j \bar{N}_p^k \bar{b}^p \\ &= \sum_{j=1}^m \sum_{p=1}^n \bar{N}_p^j a^j \bar{b}^p\end{aligned}$$

- ▶ Therefore,  $(L(v), w) = (v, L^*(w))$  for all  $v \in V$  and  $w \in W$  if and only if

$$\bar{N}_p^j = M_j^p, \text{ i.e., } N_p^j = \bar{M}_j^p,$$

or equivalently,

$$N = M^*$$

# Adjoint of Linear Maps and Matrices (Part 5)

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- ▶ If  $E$  is a unitary basis of  $V$  and  $F$  is a unitary basis of  $W$ ,  $L : V \rightarrow W$  is a linear map, and  $M$  is a matrix that satisfies

$$L(E) = FM,$$

then

$$L^*(F) = EM^*$$

# Basic Properties of Adjoint Map

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► If  $L, L_1, L_2 : V \rightarrow W$  are linear maps and  $c \in \mathbb{F}$ , then

$$(L_1 + L_2)^* = L_1^* + L_2^*$$

$$(cL)^* = \bar{c}L^*$$

$$(L_1 \circ L_2)^* = L_2^* \circ L_1^*$$

$$(L^*)^* = L$$

$$(w, L(v)) = (L^*(w), v)$$