

MA-GY 7043: Linear Algebra II

Course Requirements

Notation

Abstract Vector Spaces

Abstract Matrices

Change of Basis

Linear Functions and Maps

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Course
Requirements
Notation
Abstract Vector
Spaces
Abstract Matrix
Notation
Change of Basis
Linear Functions
and Maps

Outline I

Course
Requirements

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Course Requirements

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Abstract Matrix Notation

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Assignments

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- ▶ All homework assignments and exams will be handled using Gradescope
- ▶ Homework
 - ▶ Every one or two weeks
 - ▶ Provided as Overleaf project and Gradescope assignment
 - ▶ Solutions must be typed up using LaTeX
 - ▶ Submissions uploaded as PDF to Gradescope
- ▶ Midterm and Final
 - ▶ In person
 - ▶ Format to be determined
 - ▶ 150 minute written exam
 - ▶ 30 minute oral exam

Grading Policy

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- ▶ Course grade
 - ▶ Homework: 20%
 - ▶ Midterm: 30%
 - ▶ Final: 50%
 - ▶ Tweaks
- ▶ Homework and Exams
 - ▶ Partial credit for correct and relevant logical reasoning
 - ▶ Full credit for correct and relevant logical reasoning and correct answer
 - ▶ No credit for correct answer but incorrect logical reasoning
 - ▶ Incorrect logic and calculations will be severely penalized

Course Information

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► Web Pages

- [My homepage](#)
- [Course Homepage](#)
- [Course Calendar](#)

► Textbook

- Yisong Yang, **A Concise Text on Advanced Linear Algebra**, Cambridge University Press
- PDF available in [Ed Discussion Resources](#)

Functions and Maps

- ▶ We will use the following notation when defining a function or map:

$$function : domain \rightarrow codomain$$

$$input \mapsto output$$

- ▶ When doing calculations and proofs, It is important to keep track of the domain and codomain of a function
- ▶ Given maps $F : X \rightarrow Y$ and $G : W \rightarrow Z$, then F can be composed with G ,

$$G \circ F : X \rightarrow Z$$

if and only if $Y \subset W$,

- ▶ If you make sure that each input to a function really is an element of the domain and each output really is treated as an element of the codomain, you will catch 90% of your errors

Logical Symbols

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- ▶ $\forall$ means *for each* or *for any* or *for all*
- ▶ $\exists$ means *there is at least one* or *there exists at least one*
- ▶ $\exists!$ means *there is exactly one* or *there exists exactly one*
- ▶ $(\text{assumption}) \implies (\text{conclusion})$ means
 - ▶ *if (assumption), then (conclusion)*
 - ▶ *(assumption) only if (conclusion)*
 - ▶ *(conclusion) if (assumption)*
- ▶ $\iff$ means *if and only if*

Abstract Vector Space

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- ▶ Let $\mathbb{F}$ be either the reals (denoted $\mathbb{R}$) or the complex numbers (denoted $\mathbb{C}$)
- ▶ A vector space over $\mathbb{F}$ is a set V with the following:
 - ▶ An element called the **zero vector**, denoted $\vec{0}$, 0_V , or simply 0
 - ▶ An operation called **vector addition**:

$$V \times V \rightarrow V$$

$$(v_1, v_2) \mapsto v_1 + v_2$$

- ▶ An operation called **scalar multiplication**:

$$V \times \mathbb{F} \rightarrow V$$

$$(v, r) \mapsto rv = vr$$

such that the following properties hold

Properties of Vector Addition

► Associativity

$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$$

► Commutativity

$$v_1 + v_2 = v_2 + v_1$$

► Identity element:

$$v + \vec{0} = v$$

► Inverse element: For each $v \in V$, there exists an element, denoted $-v$, such that

$$v + (-v) = \vec{0}$$

Scalar Multiplication

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► Properties

► Associativity

$$(f_1 f_2)v = f_1(f_2 v)$$

► Distributivity

$$(f_1 + f_2)v = f_1 v + f_2 v$$

$$f(v_1 + v_2) = f v_1 + f v_2$$

► Identity element

$$1v = v$$

Consequences



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$$\begin{aligned}0v &= 0v + v - v \\&= 0v + 1v - v \\&= (0 + 1)v - v \\&= v - v \\&= \vec{0}\end{aligned}$$

$$\begin{aligned}(-1)v &= (-1)v + v - v \\&= (-1)v + 1v - v \\&= (-1 + 1)v - v \\&= 0v - v \\&= \vec{0} - v \\&= -v\end{aligned}$$

Valid and Invalid Expressions

► Valid expressions

$(\text{vector}) + (\text{vector})$

$(\text{scalar}) + (\text{scalar})$

$(\text{scalar})(\text{vector})$

$(\text{vector})(\text{scalar})$

$(\text{scalar})(\text{scalar})$

► Invalid expressions

$(\text{vector}) + (\text{scalar})$

$(\text{scalar}) + (\text{vector})$

$(\text{vector})(\text{vector})$

Linear Combination of Vectors

- ▶ Given a finite set of vectors $v_1, \dots, v_m \in V$ and scalars $f^1, \dots, f^m$, the vector

$$f^1 v_1 + \dots + f^m v_m$$

is called a **linear combination** of $v_1, \dots, v_m$

- ▶ Given a subset $S \subset V$, not necessarily finite, the **span** of S is the set of all possible linear combinations of vectors in S

$$[S] =$$

$$\{f^1 v_1 + \dots + f^m v_m : \forall f^1, \dots, f^m \in \mathbb{F} \text{ and } v_1, \dots, v_m \in S\}$$

- ▶ A vector space V is **finite dimensional** if there is a finite set S of vectors such that

$$[S] = V$$

- ▶ *In this course vector spaces are assumed to be finite dimensional*

Basis of a Vector Space

- ▶ A set $\{v_1, \dots, v_k\} \subset V$ is **linearly independent** if

$$f^1 v_1 + \dots + f^m v_m = \vec{0} \implies f^1 = \dots = f^m = 0,$$

- ▶ A finite set $S = (v_1, \dots, v_m) \subset V$ is called a **basis** of V if it is linearly independent and

$$[S] = V$$

- ▶ For such a basis, if $v \in V$, then there exist a unique set of scalar coefficients $(a^1, \dots, a^m)$ such that

$$v = a^k v_k$$

- ▶ In other words, the map

$$\begin{aligned} \mathbb{F}^m &\rightarrow V \\ \langle f^1, \dots, f^m \rangle &\mapsto f^1 v_1 + \dots + f^m v_m \end{aligned}$$

is bijective

Dimension of a Vector Space

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- ▶ Every finite dimensional vector space has a basis
- ▶ Any two bases have the same number of elements
- ▶ The dimension of a vector space is defined to be the number of elements in a basis
- ▶ The dimension of V is denoted $\dim V$

Definition of an Abstract Matrix

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- ▶ An m -by- n **abstract matrix** M is a table of symbols with m rows and n columns
- ▶ The element in the j -th row and k -th column is labeled

$$M_k^j$$

- ▶ Therefore,

$$M = \begin{bmatrix} M_1^1 & \cdots & M_n^1 \\ \vdots & & \vdots \\ M_1^m & \cdots & M_n^m \end{bmatrix}$$

Row and Column Matrices

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- ▶ A **row matrix** is a matrix with 1 row,

$$R = [R_1 \quad \cdots \quad R_n]$$

- ▶ A **column matrix** is a matrix with 1 column

$$C = \begin{bmatrix} C^1 \\ \vdots \\ C^m \end{bmatrix}$$

Product of Row and Column Matrices (Part 1)

- Let R be a row matrix with m columns and C be a column matrix with m rows,

$$R = [R_1 \quad \cdots \quad R_m] \quad \text{and} \quad C = \begin{bmatrix} C^1 \\ \vdots \\ C^m \end{bmatrix}$$

- Suppose that for each $1 \leq k \leq m$, the product

$$R_j C^j$$

is well defined, e.g.,

$$R_1, \dots, R_m, C^1, \dots, C^m \in \mathbb{F} \quad (1)$$

$$R_1, \dots, R_m \in V \text{ and } C^1, \dots, C^m \in \mathbb{F} \quad (2)$$

$$R_1, \dots, R_m \in \mathbb{F} \text{ and } C^1, \dots, C^m \in V \quad (3)$$

Product of Row and Column Matrices (Part 2)

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- ▶ The **matrix product** of R and C is defined to be the 1-by-1 matrix

$$RC = \begin{bmatrix} R_1 & \cdots & R_m \end{bmatrix} \begin{bmatrix} C^1 \\ \vdots \\ C^m \end{bmatrix} = R_1 C^1 + \cdots + R_m C^m$$

- ▶ If (1) holds, then RC is a scalar-valued 1-by-1 matrix
- ▶ If (2) or (3) holds, then RC is a vector-valued 1-by-1 matrix

Product of Two Matrices

- ▶ Let $R^1, \dots, R^m$ denote the rows of an m -by- k matrix

$$M = \begin{bmatrix} M_1^1 & \cdots & M_k^1 \\ \vdots & & \vdots \\ M_1^m & \cdots & M_k^m \end{bmatrix} = \begin{bmatrix} R^1 \\ \vdots \\ R^m \end{bmatrix}$$

- ▶ Let $C_1, \dots, C_n$ denote the columns of a k -by- n matrix

$$N = \begin{bmatrix} N_1^1 & \cdots & N_n^1 \\ \vdots & & \vdots \\ N_1^k & \cdots & N_n^k \end{bmatrix} = [C_1 \quad \cdots \quad C_n]$$

- ▶ The product of M and N is defined to be the m -by- n matrix, denoted MN , where for each

$$1 \leq j \leq m \text{ and } 1 \leq k \leq n,$$

the element in the j -th row and k -th column is

$$(MN)_k^j = R^j C_k$$

Properties of Abstract Matrix Multiplication

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- ▶ If A, B are m -by- k matrices and C is a k -by- n matrix, then

$$(A + B)C = AC + BC$$

- ▶ If A is an m -by- k matrix and B, C are k -by- n matrices, then

$$A(B + C) = AB + AC$$

- ▶ If A is an m -by- j matrix, B is a j -by- k matrix, and C is a k -by- n matrix, then

$$(AB)C = A(BC)$$

Matrix Notation for Vector with Respect to Basis

- ▶ Let $(b_1, \dots, b_m)$ be a basis of a vector space V
- ▶ For each $v \in V$, there are unique coefficients $c^1, \dots, c^m \in \mathbb{F}$ such that

$$\begin{aligned} v &= b_1 c^1 + \dots + b_m c^m \\ &= \begin{bmatrix} b_1 & \dots & b_m \end{bmatrix} \begin{bmatrix} c^1 \\ \vdots \\ c^m \end{bmatrix} \\ &= BC, \end{aligned}$$

where the basis is written as a row matrix of vectors

$$B = \begin{bmatrix} b_1 & \dots & b_m \end{bmatrix}$$

and the coefficients are written as a column matrix of scalars

$$C = \begin{bmatrix} c^1 \\ \vdots \\ c^m \end{bmatrix}$$

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Matrices of Matrices

- ▶ Let M be an abstract m -by- k matrix

$$M = \begin{bmatrix} M_1^1 & \cdots & M_k^1 \\ \vdots & & \vdots \\ M_1^m & \cdots & M_k^m \end{bmatrix}$$

where each M_j^i is itself an p -by- p matrix

- ▶ Therefore, M is an mp -by- kp matrix, broken up into p -by- p blocks
- ▶ Let N be an abstract k -by- n matrix

$$N = \begin{bmatrix} N_1^1 & \cdots & N_n^1 \\ \vdots & & \vdots \\ N_1^k & \cdots & N_n^k \end{bmatrix}$$

where each N_l^j is itself an p -by- p matrix

- ▶ Then the abstract matrix product $A = MN$ is the same as the standard matrix product $A = MN$

Change of Basis of Formula

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- ▶ Let $E = (e_1, \dots, e_n)$ be a basis of V and

$$v = a^1 e_1 + \dots + a^n e_n$$

- ▶ If $F = (f_1, \dots, f_n)$ is another basis, then there is a unique matrix M such that for each $1 \leq k \leq n$,

$$f_k = M_k^1 e_1 + \dots + M_k^n e_n$$

- ▶ v can also be written with respect to the basis F ,

$$v = b^1 f_1 + \dots + b^n f_n$$

- ▶ How are $(a^1, \dots, a^n)$ and $(b^1, \dots, b^n)$ related?

Standard Basis of $\mathbb{F}^3$

- Denote the standard basis vectors of $\mathbb{F}^3$ by

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- The basis can be written as a row matrix of column vectors:

$$E = [e_1 \quad e_2 \quad e_3] = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] = I$$

- Any vector $v = (v^1, v^2, v^3) \in \mathbb{F}$ can be written as

$$v = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = e_1 v^1 + e_2 v^2 + e_3 v^3 = [e_1 \quad e_2 \quad e_3] \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = Ev$$

Change of Basis Example on $\mathbb{F}^3$

- Consider a basis of $\mathbb{F}^3$,

$$F = [f_1 \quad f_2 \quad f_3] = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{array} \right]$$

- Given a vector $v = (v^1, v^2, v^3)$, there are coefficients b^1, b^2, b^3 such that

$$\begin{aligned} v = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} &= f_1 b^1 + f_2 b^2 + f_3 b^3 \\ &= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} b^1 + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} b^2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} b^3 = Fb \end{aligned}$$

- Therefore,

$$b = F^{-1}v$$

Change of Basis Example on $\mathbb{F}^3$

- Consider a basis

$$F = [f_1 \quad f_2 \quad f_3] = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{array} \right]$$

- Given a vector $v = (1, 2, 3)$, there are coefficients b^1, b^2, b^3 such that

$$\begin{aligned} (1, 2, 3) &= b^1(1, -1, 1) + b^2(0, 1, 1) + b^3(0, 0, 1) \\ &= (b^1, -b^1 + b^2, b^1 + b^3 + b^3) \end{aligned}$$

or, equivalently,

$$\begin{aligned} b^1 &= 1 \\ -b^1 + b^2 &= 2 \\ b^1 + b^2 + b^3 &= 3 \end{aligned}$$

- Unique solution is $(b^1, b^2, b^3) = (1, 3, -1)$

Change of Basis on Abstract Vector Space

- ▶ Consider two different bases of an n -dimensional vector space V ,

$$E = [e_1 \quad \cdots \quad e_n] \quad \text{and} \quad F = [f_1 \quad \cdots \quad f_n]$$

- ▶ Since E is a basis, we can write each basis vector of F as a linear combination of the vectors in E

$$\begin{aligned} F &= [f_1 \mid \cdots \mid f_n] \\ &= [e_1 M_1^1 + \cdots + e_n M_1^n \mid \cdots \mid e_1 M_n^1 + \cdots + e_n M_n^n] \\ &= [e_1 \quad \cdots \quad e_n] \begin{bmatrix} M_1^1 & \cdots & M_n^1 \\ \vdots & & \vdots \\ M_1^n & \cdots & M_n^n \end{bmatrix} \\ &= EM, \end{aligned}$$

where M is a square matrix of scalars

Change of Coefficients

- ▶ Any vector v can be written as either a linear combination of the basis E ,

$$v = e_1 a^1 + \cdots + e_n a^n = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix} = E a$$

or as a linear combination of the basis F ,

$$v = f_1 b^1 + \cdots + f_n b^n = \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix} \begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix} = F b$$

- ▶ Since $F = EM$,

$$v = Fb = (EM)b = E(Mb)$$

- ▶ Therefore,

$$a = Mb \text{ and } b = M^{-1}a$$

Change of Basis Formula

- ▶ Let E and F be bases of V such that

$$F = EM,$$

- ▶ If $v = Ea = Fb$, then

$$a = Mb \text{ and } b = M^{-1}a$$

- ▶ The matrix that transforms old coefficients into new coefficients is the inverse of the matrix that transforms the old basis into the new basis
- ▶ Equivalently, the matrix that transforms the old basis into the new basis is the matrix that transforms the new coefficients into the old coefficients
- ▶ **WARNING:** This works only if you write a basis as a row matrix of vectors and the coefficients as a column matrix of scalars

Linear Functions

- If V is a vector space, then a function

$$\ell : V \rightarrow \mathbb{F}$$

is **linear**, if for any $v_1, v_2 \in V$

$$\ell(v_1 + v_2) = \ell(v_1) + \ell(v_2)$$

and for any $v \in V$ and $s \in \mathbb{F}$,

$$\ell(vs) = \ell(v)s$$

- Consequences:

$$\ell(0_V) = 0$$

$$\ell(-v) = -\ell(v)$$

Properties of Linear Functions

- ▶ If ℓ_1, ℓ_2 are linear functions, then so is $\ell_1 + \ell_2$
- ▶ If 0 is the zero function, it is linear and for any linear function ℓ ,

$$\ell + 0 = \ell$$

- ▶ If $s \in \mathbb{F}$ and ℓ is a linear function, then the function $s\ell$, which is defined by

$$(s\ell)(v) = s(\ell(v)),$$

is also a linear function

- ▶ If we denote $-\ell = (-1)\ell$, then

$$\ell + (-\ell) = 0$$

- ▶ It is straightforward to verify that these operations satisfy the properties of vector addition and scalar multiplication
- ▶ It follows that the set of all linear functions on V , denoted V^* , is a vector space
- ▶ It is called the **dual vector space** of V

Linear Maps

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- If V and W are vector spaces, then

$$L : V \rightarrow W$$

is a **linear map**, if for any $v, v_1, v_2 \in V$ and $s \in \mathbb{F}$,

$$L(v_1 + v_2) = L(v_1) + L(v_2)$$

$$L(sv) = sL(v)$$

- Consequences:

$$L(0_V) = 0_W$$

$$L(-v) = -L(v)$$

Properties of Linear Maps

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- ▶ If $K : U \rightarrow V$ and $L : V \rightarrow W$ are linear maps, then so is

$$L \circ K : U \rightarrow W$$

- ▶ If $L : V \rightarrow W$ is bijective, it is called a **linear isomorphism**
- ▶ If $L : V \rightarrow W$ is a linear isomorphism, then so is

$$L^{-1} : W \rightarrow V$$

n -Dimensional Vector Spaces are Isomorphic

- ▶ Let $\dim V = \dim W = m$
- ▶ Let $E = (e_1, \dots, e_m)$ be a basis of V
- ▶ Let $F = (f_1, \dots, f_m)$ be a basis of W
- ▶ The map

$$\begin{aligned} L_{E,F} : V &\rightarrow W \\ e_1 a^1 + \dots + e_m a^m &\mapsto f_1 a^1 + \dots + f_m a^m \end{aligned}$$

is a linear isomorphism

- ▶ Given any basis $(e_1, \dots, e_m)$ of V , there is a linear isomorphism

$$\begin{aligned} L_V : \mathbb{F}^m &\rightarrow V \\ (a^1, \dots, a^m) &\mapsto e_1 a^1 + \dots + e_m a^m \end{aligned}$$

Space of Linear Maps

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- ▶ Let $\text{Hom}(V, W)$ denote the set of all linear maps with domain V and codomain W
- ▶ It is straightforward to check that if $L_1, L_2, L \in \text{Hom}(V, W)$ and $s \in \mathbb{F}$, then

$$L_1 + L_2, sL \in \text{Hom}(V, W)$$

are also linear maps from V to W

- ▶ It is also easily checked that these operations satisfy the properties of vector addition and scalar multiplication
- ▶ It follows that $\text{Hom}(V, W)$ is itself also a vector space
- ▶ Observe that $V^* = \text{Hom}(V, \mathbb{F})$

Endomorphisms and Automorphisms

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- ▶ The space of **endomorphisms** of V is defined to be

$$\text{End}(V) = \text{Hom}(V, V)$$

- ▶ An endomorphism $L : V \rightarrow V$ is an **automorphism** if it is bijective
- ▶ The space of automorphisms of V is denoted $\text{Aut}(V)$

Linear maps from $\mathbb{F}^m$ to $\mathbb{F}^n$

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- ▶ Let $\text{gl}(n, m, \mathbb{F})$ denote the vector space of n -by- m matrices with components in $\mathbb{F}$
 - ▶ $\dim \text{gl}(n, m, \mathbb{F}) = nm$
 - ▶ Let $\text{gl}(n, \mathbb{F}) = \text{gl}(n, n, \mathbb{F})$
 - ▶ Let $\text{gl}(n) = \text{gl}(n, \mathbb{R})$
- ▶ There is a natural linear isomorphism

$$\text{Hom}(\mathbb{F}^m, \mathbb{F}^n) = \text{gl}(n, m, \mathbb{F})$$

Matrix as Linear Map

- ▶ Let $E = (e_1, \dots, e_m)$ be a basis of V
- ▶ Let $F = (f_1, \dots, f_n)$ be a basis of W
- ▶ For each $M \in \text{gl}(n, m, \mathbb{F})$, let $L : V \rightarrow W$ be the linear map where

$$\forall 1 \leq k \leq m, L(e_k) = f_1 M_k^1 + \dots + f_n M_k^n$$

and therefore for any $v = e_1 a^1 + \dots + e_m a^m = Ea$,

$$\begin{aligned} L(v) &= L(e_1 a^1 + \dots + e_m a^m) \\ &= L(e_1) a^1 + \dots + L(e_m) a^m \\ &= (f_1 M_1^1 + \dots + f_n M_1^n) a^1 + \dots + (f_1 M_m^1 + \dots + f_n M_m^n) a^m \\ &= f_1 (M_1^1 a^1 + \dots + M_m^1 a^m) + \dots + f_n (M_1^n a^1 + \dots + M_m^n a^m) \\ &= f_1 (Ma)^1 + \dots + f_n (Ma)^n \end{aligned}$$

- ▶ This defines a map $l_{E,F} : \text{gl}(n, m, \mathbb{F}) \rightarrow \text{Hom}(V, W)$

Linear Map as Matrix

- ▶ Let $E = (e_1, \dots, e_m)$ be a basis of V
- ▶ Let $F = (f_1, \dots, f_n)$ be a basis of W
- ▶ Let $L : V \rightarrow W$ be a linear map
- ▶ For each e_k , $1 \leq k \leq m$, there exists $(M_k^1, \dots, M_k^n) \in \mathbb{F}^n$ such that

$$L(e_k) = f_1 M_k^1 + \dots + f_n M_k^n$$

- ▶ Therefore, for any $v = e_1 a^1 + \dots + e_m a^m \in V$,

$$\begin{aligned} L(v) &= L(e_1 a^1 + \dots + e_m a^m) \\ &= L(e_1) a^1 + \dots + L(e_m) a^m \\ &= (f_1 M_1^1 + \dots + f_n M_1^n) a^1 + \dots + (f_1 M_m^1 + \dots + f_n M_m^n) a^m \\ &= f_1 (M_1^1 a^1 + \dots + M_m^1 a^m) + \dots + f_n (M_1^n a^1 + \dots + M_m^n a^m) \\ &= f_1 (Ma)^1 + \dots + f_n (Ma)^n \end{aligned}$$

- ▶ This defines a map $J_{E,F} : \text{Hom}(V, W) \rightarrow \text{gl}(n, m, \mathbb{F})$
- ▶ $J_{E,F} = I_{E,F}^{-1}$ and $I_{E,F} = J_{E,F}^{-1}$
- ▶ Therefore, $\dim \text{Hom}(V, W) = \dim \text{gl}(n, m, \mathbb{F}) = nm$

Concrete to Abstract Notation

Course
Requirements

Notation

Abstract Vector
Spaces

Abstract Matrix
Notation

Change of Basis

Linear Functions
and Maps

$$\begin{aligned} L(v) &= L(e_1 a^1 + \cdots + e_m a^m) = L \left(\begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix} \right) \\ &= L \left(\begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} \right) \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix} = \begin{bmatrix} L(e_1) & \cdots & L(e_m) \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix} \\ &= \begin{bmatrix} f_1 M_1^1 + \cdots + f_n M_1^n & \cdots & f_1 M_n^1 + \cdots + f_n M_n^n \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix} \\ &= \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix} \begin{bmatrix} M_1^1 & \cdots & M_m^1 \\ \vdots & & \vdots \\ M_1^n & \cdots & M_m^n \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix} = FMa \end{aligned}$$

Change of Basis Formula for Linear Map

- ▶ Let $E = (e_1, \dots, e_m)$ be a basis of V
- ▶ Let $F = (f_1, \dots, f_m)$ be another basis of V
- ▶ There is a matrix B such that $F = EB$, i.e.,

$$f_k = e_j B_k^j$$

- ▶ Consider a linear map $L : V \rightarrow V$
- ▶ There is a matrix M such that

$$L(e_k) = e_j M_k^j, \text{ i.e., } L(E) = EM$$

and a matrix N such that

$$L(f_k) = f_j N_k^j, \text{ i.e., } L(F) = FN$$

- ▶ It follows that

$$FN = L(F) = L(EB) = L(E)B = EMB = FB^{-1}MB$$

Change of Basis Formula for Matrix

- ▶ Let $E = (e_1, \dots, e_m)$ be the standard basis of $\mathbb{F}^m$
- ▶ Let $F = (f_1, \dots, f_m)$ be another basis of $\mathbb{F}^m$
- ▶ There is a matrix B such that $F = EB$, i.e.,

$$f_k = e_j B_k^j$$

- ▶ Consider a m -by- m matrix M and the corresponding map

$$L : \mathbb{F}^m \rightarrow \mathbb{F}^m$$

$$L(v) = (v^1, \dots, v^m) = Mv = EMv$$

- ▶ It follows that with respect to the basis F , if

$$Ev = v = Fw = EBw,$$

then $v = Bw$ and

Composition is Matrix Multiplication

- ▶ Consider vector spaces U, V, W and linear maps

$$K : U \rightarrow V, L : V \rightarrow W$$

- ▶ Let $(e_1, \dots, e_k)$ be a basis of U
- ▶ Let $(f_1, \dots, f_m)$ be a basis of V
- ▶ Let $(g_1, \dots, g_n)$ be a basis of W
- ▶ There is an m -by- k matrix M such that

$$K(e_j) = f_p M_j^p, \quad 1 \leq j \leq k$$

- ▶ There is an n -by- m matrix N such that

$$L(f_p) = g_a N_p^a, \quad 1 \leq p \leq m$$

- ▶ There is an n -by- k matrix P such that

$$(L \circ K)(e_j) = g_a P_j^a, \quad 1 \leq j \leq k$$

- ▶ On the other hand,

$$(L \circ K)(e_j) = L(K(e_j)) = L(f_p M_j^p) = L(f_p) M_j^p = g_a N_p^a M_j^p$$

- ▶ Therefore, $P_j^a = N_p^a M_j^p$.