

MA-GY7033 Linear Algebra I

Determinant of Matrix
Determinant of Linear Endomorphism
Determinant of Linear Map

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Outline

1. Determinant of Matrix
2. Determinant of Linear Endomorphism

Determinant of Matrix

- ▶ Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbb{R}^n$
- ▶ There is a unique antisymmetric multilinear function

$$D : \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}$$

with n inputs such that

$$D(e_1, \dots, e_n) = 1,$$

- ▶ The formula for D is

$$D(v_1, \dots, v_n) = \sum_{\phi \in T_n} \epsilon(\phi) b_{\phi(1)}^1 \cdots b_{\phi(n)}^n = \sum_{\sigma \in S_n} \epsilon(\sigma) b_{\sigma(1)}^1 \cdots b_{\sigma(n)}^n$$

where each $v_k = (b_k^1, \dots, b_k^n)$

- ▶ Let M be an n -by- n matrix with columns $C_1, \dots, C_n$
- ▶ The determinant of M is defined to be

$$\det M = D(C_1, \dots, C_n)$$

Formula and Characterization of the Determinant

- ▶ The determinant function $\det : \mathcal{M}_{n \times n}(\mathbb{R}) \rightarrow \mathbb{R}$ is the unique function such that
 - ▶ $\det M$ is an antisymmetric multilinear function of the columns of M
 - ▶ $\det I = 1$

- ▶ If

$$M = \begin{bmatrix} M_1^1 & \cdots & M_n^1 \\ \vdots & & \vdots \\ M_1^n & \cdots & M_n^n \end{bmatrix},$$

then

$$\det M = \sum_{\phi \in T_n} \epsilon(\phi) M_{\phi(1)}^1 \cdots M_{\phi(n)}^n = \sum_{\sigma \in S_n} \epsilon(\sigma) M_{\sigma(1)}^1 \cdots M_{\sigma(n)}^n$$

- ▶ This formula can be useful in proofs
- ▶ To calculate the determinant of a specific matrix M , it is usually easier to use the properties of an antisymmetric multilinear function

Multiplicative Property of the Determinant

A fundamental property of the determinant is that if A and B are n -by- n matrices, then

$$\det AB = (\det A)(\det B)$$

Proof of Multiplicative Property

- ▶ Given a matrix A , define $D_A : (\mathbb{R}^n)^n \rightarrow \mathbb{R}$ by

$$D_A(v_1, \dots, v_n) = \det(AM),$$

where $v_1, \dots, v_n$ are the columns of M .

- ▶ If $C_1, \dots, C_n$ are the columns of B , then the columns of AB are $AC_1, \dots, AC_n$.
- ▶ The function D_A is alternating and multilinear, so

$$\begin{aligned}\det(AB) &= D_A(C_1, \dots, C_n) \\ &= D_A(e_1, \dots, e_n) D(C_1, \dots, C_n) \\ &= (\det A)(\det B).\end{aligned}$$

M Invertible $\implies \det M \neq 0$

- ▶ If M is invertible, then

$$(\det M)(\det M^{-1}) = \det(MM^{-1}) = \det I = 1.$$

- ▶ Therefore,

$$\det M \neq 0, \quad \det M^{-1} = \frac{1}{\det M}.$$

- ▶ Contrapositively, if $\det M = 0$, then M is not invertible.

M Singular $\iff \det M = 0$

- There are bases E and F such that

$$ME = FI_k,$$

where $k = \text{rank } M$ and I_k has k ones on the diagonal and all other entries zero.

- Hence

$$M = FI_kE^{-1}.$$

- Therefore,

$$\det M = 0 \iff \text{rank } M < n \iff M \text{ is singular.}$$

Transpose of a Matrix

- ▶ If $M \in \mathcal{M}_{m \times n}$, its transpose $M^T \in \mathcal{M}_{n \times m}$ switches rows and columns:

$$(M^T)_k^j = M_j^k.$$

- ▶ In matrix notation,

$$\begin{bmatrix} M_1^1 & \cdots & M_m^1 \\ \vdots & & \vdots \\ M_1^n & \cdots & M_m^n \end{bmatrix}^T = \begin{bmatrix} M_1^1 & \cdots & M_1^m \\ \vdots & & \vdots \\ M_n^1 & \cdots & M_n^m \end{bmatrix}.$$

- ▶ If $A \in \mathcal{M}_{k \times m}$ and $B \in \mathcal{M}_{m \times n}$, then

$$(AB)^T = B^T A^T \in \mathcal{M}_{n \times k}.$$

Determinant of Matrix Equals Determinant of Its Transpose

- ▶ Given any square matrix M ,

$$\det M^T = \det M$$

- ▶ Proof 1: Use the formula for the determinant

$$\begin{aligned}\det M &= \sum_{\sigma \in S_n} \epsilon(\sigma) M_1^{\sigma(1)} \cdots M_n^{\sigma(n)} \\ &= \sum_{\sigma \in S_n} \epsilon(\sigma) M_{\sigma^{-1}(1)}^1 \cdots M_{\sigma^{-1}(n)}^n \\ &= \sum_{\sigma \in S_n} \epsilon(\sigma) M_{\sigma(1)}^1 \cdots M_{\sigma(n)}^n \\ &= \det M^T\end{aligned}$$

Determinant of the Transpose: A Second Proof

- ▶ Write $M = PLU$, where
 - ▶ P is a permutation matrix,
 - ▶ L is lower triangular, and
 - ▶ U is upper triangular.
- ▶ Transposition preserves the determinant of triangular and permutation matrices:

$$\det L^T = \det L, \quad \det U^T = \det U, \quad \det P^T = \det P.$$

- ▶ Therefore,

$$\begin{aligned}\det M^T &= \det(U^T L^T P^T) \\ &= (\det U)(\det L)(\det P) \\ &= \det(PLU) = \det M.\end{aligned}$$

Solving a system of n linear equations with n unknowns

- ▶ Consider a system of equations

$$a_1^1 x^1 + \cdots + a_n^1 x^n = b^1$$

$$\vdots$$

$$a_1^n x^1 + \cdots + a_n^n x^n = b^n$$

- ▶ This can be written as

$$Ax = b$$

- ▶ If $\det A \neq 0$, then given $b \in \mathbb{R}^n$, there is a unique solution

$$x = A^{-1}b$$

- ▶ If $\det A = 0$, then

- ▶ $\dim \ker A > 0$ and $\dim A(\mathbb{R}^n) < n$
- ▶ A solution exists only if $b \in A(\mathbb{R}^n) \subsetneq \mathbb{R}^n$
- ▶ If a solution exists, there are infinitely many, because

$$Ax = b \text{ and } v \in \ker A \implies A(x + v) = b$$

Pullback of multilinear function

- ▶ Consider a function

$$A : W \times \cdots \times W \rightarrow \mathbb{F},$$

- ▶ Given a map $L : V \rightarrow W$, the **pullback** of A by L is defined to be the function

$$L^*A : V \times \cdots V \rightarrow \mathbb{F}$$

given by

$$L^*A(v_1, \dots, v_k) = A(L(v_1), \dots, L(v_k))$$

- ▶ If L is linear, then
 - ▶ A multilinear implies L^*A multilinear
 - ▶ A antisymmetric implies L^*A antisymmetric
- ▶ Let $\Lambda^k V^*$ denote the vector space of antisymmetric k -linear functions on V
- ▶ Given any linear map $L : V \rightarrow W$, there is a naturally associated map

$$L^* : \Lambda^k W^* \rightarrow \Lambda^k V^*$$

- ▶ $\Lambda^k W^*, \Lambda^k V^*$ are vector spaces and L^* is a linear map

Oriented Volume Functions

- ▶ If $n = \dim(V)$, $\Lambda^n V^*$ is the vector space of all antisymmetric n -linear functions on V
- ▶ Let $A \in \Lambda^n V^* \setminus \{0\}$
- ▶ Recall that for any $B \in \Lambda^n V^*$, there is a scalar $b \in \mathbb{F}$ such that

$$B = cA$$

- ▶ In particular, if $(e_1, \dots, e_n)$ is a basis, then

$$b = \frac{B(e_1, \dots, e_n)}{A(e_1, \dots, e_n)}$$

- ▶ This implies that $\dim(\Lambda^n V^*) = 1$

Determinant of Linear Endomorphism

- ▶ If U is a 1-dimensional vector space and $F : U \rightarrow U$ is linear, then there exists a scalar c such that for any $u \in U$,

$$F(u) = cu$$

- ▶ If $L : V \rightarrow V$ is a linear map, then the pullback of L is a linear map

$$L^* : \Lambda^n V^* \rightarrow \Lambda^n V^*$$

- ▶ Since $\dim(\Lambda^n V^*) = 1$, there exists a scalar $c \in \mathbb{F}$ such that for any $A \in \Lambda^n V^*$,

$$L^*A = \delta A$$

- ▶ The determinant of L is defined to be δ
- ▶ I.e., the determinant of L is the unique scalar that satisfies

$$L^*A = (\det(L))A$$

for any $A \in \Lambda^n V^*$

Computation of Determinant

- ▶ Let (e_1, e_2, e_3) be a basis of V
- ▶ Let $L : V \rightarrow V$ be the linear map given by

$$L(e_1) = 2e_1 + 3e_2 + 4e_3$$

$$L(e_2) = 5e_2 - 6e_3$$

$$L(e_3) = 7e_3$$

- ▶ Given any $A \in \Lambda^3 V^* \setminus \{0\}$,

$$\begin{aligned} L^*A(e_1, e_2, e_3) &= A(L(e_1), L(e_2), L(e_3)) \\ &= A(2e_1 + 3e_2 + 4e_3, 5e_2 - 6e_3, 7e_3) \\ &= A(2e_1 + 3e_2 + 4e_3, 5e_2, 7e_3) \\ &= A(2e_1, 5e_2, 7e_3) \\ &= 70A(e_1, e_2, e_3) \end{aligned}$$

- ▶ It follows that $\det(L) = 70$

Determinant of Sparse Matrix

- Suppose we want to calculate

$$\det \begin{pmatrix} \begin{bmatrix} 2 & 0 & -1 & 0 \\ 1 & 3 & 1 & 5 \\ 0 & 0 & 3 & 0 \\ 2 & -2 & 1 & 0 \end{bmatrix} \end{pmatrix}$$

- This is equal to

$$\begin{aligned} & D(Me_1, Me_2, Me_3, Me_4) \\ &= D(2e_1 + e_2 + 2e_4, 3e_2 - 2e_4, -e_1 + e_2 + 3e_3 + e_4, 5e_2) \\ &= D(2e_1 + 2e_4, -2e_4, -e_1 + 3e_3 + e_4, 5e_2) \\ &= D(2e_1, -2e_4, -e_1 + 3e_3, 5e_2) \\ &= D(2e_1, -2e_4, 3e_3, 5e_2) \\ &= -60D(e_1, e_4, e_3, e_2) \\ &= 60D(e_1, e_2, e_3, e_4) \\ &= 60 \end{aligned}$$

Multiplicative Property of Determinant

- ▶ Recall that given $D \in \Lambda^n V^* \setminus \{0\}$,

$$L^*D = \det(L)D,$$

i.e., for any $v_1, \dots, v_n \in V$,

$$D(L(v_1), \dots, L(v_n)) = \det(L)D(v_1, \dots, v_n)$$

- ▶ Given $L_1 : V \rightarrow V$, $L_2 : V \rightarrow V$, $D \in \Lambda^n V^* \setminus \{0\}$, and a basis $(e_1, \dots, e_n)$,

$$\begin{aligned}\det(L_2 \circ L_1)D(e_1, \dots, e_n) &= D(L_2 \circ L_1(e_1), \dots, L_2 \circ L_1(e_n)) \\ &= D(L_2(L_1(e_1)), \dots, L_2(L_1(e_n))) \\ &= \det(L_2)D(L_1(e_1), \dots, L_1(e_n)) \\ &= \det(L_2)D(e_1, \dots, e_n) \\ &= \det(L_2)\det(L_1)D(e_1, \dots, e_n)\end{aligned}$$

- ▶ Since $D(e_1, \dots, e_n) \neq 0$, it follows that

$$\det(L_2 \circ L_1) = \det(L_1)\det(L_2)$$

Determinant of Invertible and Singular Maps

- ▶ If $L : V \rightarrow V$ is singular (i.e., not injective), then there exists $v \neq 0$ such that $L(v) = 0$
- ▶ If we set $e_1 = v$ and extend to a basis $(e_1, \dots, e_n)$, then

$$\det(L)D(e_1, e_2, \dots, e_n) = D(L(e_1, \dots, e_n)) = 0,$$

which implies $\det(L) = 0$

- ▶ If $L : V \rightarrow V$ is a bijective linear map and $L^{-1} : V \rightarrow V$ is the inverse map, then L^{-1} is also linear and, since

$$L^{-1} \circ L = L \circ L^{-1} = I,$$

it follows that

$$\det(L^{-1})\det(L) = 1,$$

which implies $\det(L), \det(L^{-1})$ are nonzero and

$$\det(L^{-1}) = \frac{1}{\det(L)}$$

Determinant if Domain and Codomain Not Same

- ▶ If $L : V \rightarrow W$ is linear, then for each $k > 0$ there is a pullback map

$$L^* : \Lambda^k W^* \rightarrow \Lambda^k V^*,$$

where for any $A \in \Lambda^k W^*$ and $v_1, \dots, v_k$,

$$L^* A(v_1, \dots, v_k) = A(L(v_1), \dots, L(v_k))$$

- ▶ If $\dim(V) = \dim(W) = n$, then

$$L^* : \Lambda^n W^* \rightarrow \Lambda^n V^*$$

is a map from a 1-dimensional vector space to another

- ▶ Since domain and codomain are different, there is no way to write L^* as scalar multiplication
- ▶ Define the determinant of L to be the map $\det(L) = L^*$