MA-GY7033 Linear Algebra I

Kernel, Image, and Rank

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Outline

1. Kernel, Image, and Rank

Kernel, Image, Rank of a Linear Map

- Consider any linear map $P : Z \to Y$
- The **kernel** of P is defined to be

$$\ker P = \{z \in Z \ : \ P(z) = 0\} \subset Z$$

  - $\ker P$ is a subspace of Z
- The **image** of P is defined to be

$$\operatorname{im} P = \{P(z) \ : \ z \in Z\} \subset Y$$

  - Also, denoted $P(Z)$
  - $P(Z)$ is a subspace of Y
- The **rank** of P is

$$\operatorname{rank} P = \dim P(Z)$$

Example 0

- Define $Z : \mathbb{R}^2 \to \mathbb{R}^3$ to be

$$Z(x, y) = (x, y, 0), \text{ for all } (x, y) \in \mathbb{R}^2$$

- In other words,

$$Z\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- $\ker Z = \{0\}$
- $Z(\mathbb{R}^2) = \{(x, y, 0) \ : \ x, y \in \mathbb{R}\} \subset \mathbb{R}^3$
  - A basis of $Z(\mathbb{R}^2)$ is $\{Z(e_1), Z(e_2)\} = \{(1, 0, 0), (0, 1, 0)\}$
- Therefore,

$$\begin{aligned} \dim \ker Z &= 0 \\ \operatorname{rank} Z &= 2 \end{aligned}$$

Example 1

- Define $W : \mathbb{R}^2 \to \mathbb{R}^3$ to be

$$W(x, y) = (y, 0, 0), \text{ for all } (x, y) \in \mathbb{R}^2$$

- In other words,

$$W\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- $\ker W = \{(x, 0) \ : \ x \in \mathbb{R}\}$
  - A basis of $\ker W$ is $\{(1, 0)\}$
- $W(\mathbb{R}^2) = \{(y, 0, 0) \ : \ y \in \mathbb{R}\}$
  - A basis of $W(\mathbb{R}^2)$ is $\{(1, 0, 0)\}$
- Therefore,

$$\dim \ker W = 1$$

$$\operatorname{rank} W = 1$$

Example 2

- Define $U : \mathbb{R}^2 \to \mathbb{R}^3$ to be

$$U(x, y) = (0, 0, 0), \text{ for all } (x, y) \in \mathbb{R}^2$$

- In other words,

$$U\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- $\ker U = \mathbb{R}^2$
- $U(\mathbb{R}^2) = \{(0, 0, 0)\}$
- Therefore,

$$\begin{aligned} \dim \ker U &= 2 \\ \operatorname{rank} U &= 0 \end{aligned}$$

Example 3

- Define $U : \mathbb{R}^3 \to \mathbb{R}^2$ to be

$$U(x, y, z) = (y, z), \text{ for all } (x, y, z) \in \mathbb{R}^3$$

- In other words,

$$U\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- $\ker U = \{(x, 0, 0) \;:\; x \in \mathbb{R}\}$
  - A basis is $\{(1, 0, 0)\}$
- $U(\mathbb{R}^3) = \mathbb{R}^2$
- Therefore,

$$\dim \ker U = 1$$

$$\operatorname{rank} U = 2$$

Example 4

- Define $U : \mathbb{R}^3 \to \mathbb{R}^2$ to be

$$U(x, y, z) = (z, 0), \text{ for all } (x, y, z) \in \mathbb{R}^3$$

- In other words,

$$U\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- $\ker U = \{(x, y, 0) \ : \ x, y \in \mathbb{R}\}$
  - A basis is $\{(1, 0, 0), (0, 1, 0)\}$
- $U(\mathbb{R}^3) = \{(z, 0) \ : \ z \in \mathbb{R}\}$
  - A basis is $\{(1, 0)\}$
- Therefore,

$$\dim \ker U = 2$$
$$\operatorname{rank} U = 1$$

Example 5

- Define $U : \mathbb{R}^3 \to \mathbb{R}^2$ to be

$$U(x, y, z) = (0, 0), \text{ for all } (x, y, z) \in \mathbb{R}^3$$

- In other words,

$$U\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- $\ker U = \mathbb{R}^3$
- $U(\mathbb{R}^3) = \{(0, 0)\}$
- Therefore,

$$\begin{aligned} \dim \ker U &= 3 \\ \operatorname{rank} U &= 0 \end{aligned}$$

Rank-Nullity Theorem

- If $L : V \to W$ is a linear map, then

$$\dim \ker L + \operatorname{rank} L = \dim V$$

Proof of Rank Theorem

- Let V and W be finite dimensional vector spaces, where $\dim V = m$
- Let $L : V \to W$ be a linear map, where $\dim kerL = k$
- If $k > 0$, then there exists a basis of V,

$$(e_1, \ldots, e_k, e_{k+1}, \ldots, e_m)$$

  such that
  - $(e_1, \ldots, e_k)$ is a basis of $\ker L \subset V$
  - $(L(e_{k+1}), \ldots, L(e_m))$ is a basis of $L(V) \subset W$
- If $k = 0$ and $(e_1, \ldots, e_m)$ is a basis of V, then $(L(e_1), \ldots, L(e_m))$ is a basis of $L(V)$
- Therefore,

$$\begin{aligned} \dim \ker L + \operatorname{rank} L &= \dim \ker L + \dim L(V) \\ &= k + (m - k) = m \\ &= \dim V \end{aligned}$$

Fundamental Facts

- Consider a linear map $L : V \to W$
- $\dim\ker L = 0 \iff L$ is 1-to-1, i.e., injective, because

$$\begin{aligned} L(v_1) = L(v_2) &\iff L(v_2) - L(v_1) = 0 \\ &\iff L(v_2 - v_1) = 0 \\ &\iff v_2 - v_1 \in \ker L = \{0\} \\ &\iff v_2 = v_1 \end{aligned}$$

- $\operatorname{rank} L = \dim W \iff L$ is onto because

$$\operatorname{rank} L = \dim W \iff \dim L(V) = \dim W \iff L(V) \quad = W$$

- $L : V \to W$ an **isomorphism** if it is 1-to-1 and onto
  - This holds if and only if $\dim\ker L = 0$ and $\operatorname{rank} L = \dim W$
  - By the rank theorem, this implies that

$$\dim\ker L + \operatorname{rank} L = \dim V \implies \dim W = \dim V$$

- Therefore, L is an isomorphism if and only if

$$\dim V = \dim W = \operatorname{rank} L \text{ and } \dim\ker L = 0$$

Fundamental Example

- Consider the linear map $L : \mathbb{R}^m \to \mathbb{R}^n$ given by

$$L(e_1) = \cdots = L(e_k) = 0 \text{ and } L(e_{k+1}) = f_1, \ldots, L(e_m) = f_{m-k},$$

where $\begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix}$ is the standard basis of $\mathbb{R}^m$ and $\begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix}$ is the standard basis of $\mathbb{R}^n$

- In other words,

$$L\left(\left[\begin{array}{c} x_{k\times 1} \\ \hline y_{m-k\times 1} \end{array}\right]\right) = \left[\begin{array}{c|c} 0_{m-k\times k} & I_{m-k\times m-k} \\ \hline 0_{n-(m-k)\times k} & 0_{n-(m-k)\times m-k} \end{array}\right]\left[\begin{array}{c} x_{k\times 1} \\ \hline y_{m-k\times 1} \end{array}\right]$$

- Then

$$\dim\ker L = k$$
$$\operatorname{rank} L = m - k$$

- If $k = 0$, then $\dim\ker L = 0$
- If $n - (m - k) = 0$, then $\operatorname{rank} L = n$

Abstract Linear Map

- Consider linear map $L : V \to W$, where $\dim V = m$ and $\dim W = n$
- In proof of Rank Theorem, it is shown that there is a basis $E = (e_1, \ldots, e_m)$ of V such that
  - $(e_1, \ldots, e_k)$ is a basis of $\ker L$
  - $(L(e_{k+1}), \ldots, L(e_m))$ is a basis of $L(V)$
- It follows that there is a basis $(f_1, \ldots, f_n)$ such that

$$\begin{bmatrix} f_1 & \cdots & f_{m-k} \end{bmatrix} = \begin{bmatrix} L(e_{k+1}) & \ldots & L(e_m) \end{bmatrix}$$

- Therefore,

$$\begin{aligned} L(\begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix}) &= \begin{bmatrix} 0_1 & \cdots & 0_k & f_1 & \cdots & f_{m-k} \end{bmatrix} \\ &= \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix} \left[\begin{array}{c|c} 0_{m-k \times k} & I_{m-k \times m-k} \\ \hline 0_{n-(m-k) \times k} & 0_{n-(m-k) \times m-k} \end{array}\right] \end{aligned}$$

Matrix Associated to an Abstract Linear Map

Given any abstract linear map $L : V \to W$, there exists a bases

$$E = \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} \text{ of } V \text{ and } F = \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix} \text{ of } W$$

such that the matrix M satisfying

$$L(E) = FM$$

is the same as the matrix in the fundamental example

$$M = \left[\begin{array}{c|c} 0_{m-k \times k} & I_{m-k \times m-k} \\ \hline 0_{n-(m-k) \times k} & 0_{n-(m-k) \times m-k} \end{array}\right]$$

Example (Part 1)

- Consider the map $L : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$L\left(\begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} v^1 + 2v^2 + 3v^3 \\ 4v^3 \end{bmatrix}$$

- $\ker L = \{(v^1, v^2, 0) \ : \ v^1 + 2v^2 = 0\}$
- A basis of $\ker L$ is $\{(-2, 1, 0)\}$
- A basis of $\mathbb{R}^3$ is $\{(-2, 1, 0), (0, 1, 0), (0, 0, 1)\}$
- A basis of $L(\mathbb{R}^3)$ is

$$\{L(0, 1, 0), L(0, 0, 1)\} = \{(2, 0), (3, 4)\}$$

Example (Part 2)

- If

$$\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \left[\begin{array}{c|c|c} -2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \text{ and } \begin{bmatrix} f_1 & f_2 \end{bmatrix} = \left[\begin{array}{c|c} 2 & 3 \\ 0 & 4 \end{array}\right]$$

- Then

$$\begin{bmatrix} L(e_1) & L(e_2) & L(e_3) \end{bmatrix} = \begin{bmatrix} 0 & f_1 & f_2 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- And given any vector $v = e_1 a^1 + e_2 a^2 + e_3 a^3$,

$$L(v) = L(e_1)a^1 + L(e_2)a^2 + L(e_3)a^3 = f_1 a^2 + f_2 a^3 = FMa,$$

where

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Linear Map With Respect to Basis (Part 1)

- A map $L : A \to B$ is linear if

$$L(c^1 a_1 + c^2 a_2) = c^1 L(a_1) + c^2 L(a_2),\ \forall c^1, c^2 \in \mathbb{R},\ \ a_1, a_2 \in A$$

  - Implies that A and B are vector spaces
- Let $E = (e_1, \ldots, e_m)$ be a basis of A and $F = (f_1, \ldots, f_n)$ be a basis of B
  - Implies $\dim A = m$ and $\dim B = n$
- There is an n-by-m matrix M such that

$$\begin{aligned} L(e_k) &= f_1 M^1_k + \cdots + f_n M^n_k \\ &= f_j M^j_k \text{ (using Einstein notation)} \\ &= FM_k\text{, the } k\text{-th column of } M \end{aligned}$$

  - The matrix M depends on L, E, and F

Linear Map With Respect to Basis (Part 2)

- Given any vector $v \in A$, it can be written with respect to the basis E,

$$
\begin{aligned}
v &= e_1 c^1 + \cdots + e_m c^m = e_k c^k \\
&= e_k c^k \\
&= Ec
\end{aligned}
$$

- Therefore,

$$
\begin{aligned}
L(v) &= L(e_1 c^1 + \cdots + e_m c^m) \\
&= L(e_1) c^1 + \cdots + L(e_m) c^m \\
&= (f_1 M_1^1 + \cdots + f_n M_1^n) c^1 + \cdots + (f_1 M_m^1 + \cdots f_n M_m^n) c^m \\
&= f_1 (M_1^1 c^1 + \cdots M_m^1 c^m) + \cdots + f_n (M_1^n c^1 + \cdots + M_m^n c^m) \\
&= \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix} \begin{bmatrix} M_1^1 & \cdots & M_m^1 \\ \vdots & & \vdots \\ M_1^n & \cdots & M_m^n \end{bmatrix} \begin{bmatrix} c^1 \\ \vdots \\ c^m \end{bmatrix} = FMc
\end{aligned}
$$

Basis of an Infinite-Dimensional Vector Space

- You can define a basis of any vector space $\mathbb{V}$ to be a linearly independent set $S \subset \mathbb{V}$ such that any $v \in \mathbb{V}$ is a finite linear combination of elements in S,

$$v = a^1 s_1 + \cdots + a^n s_n$$

- Example: $\mathcal{P} = \{\text{polynomials in } x \text{ with real coefficients}\}$
  - $S = \{1, x, x^2, \cdots\}$
- Harder example: $C(\mathbb{R}) = \{\text{continuous functions } f : \mathbb{R} \to \mathbb{R}\}$
  - Such a basis does exist, by Zorn's lemma, but no explicit one can be written down
- New concept needed: topological vector spaces