

MA-GY7033 Linear Algebra I

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Outline

1. Course Information
2. Vector Spaces
3. Span, Independence, and Dimension

Course Web Site

- ▶ <https://www.math.nyu.edu/~dy444/MA-GY7033Fall2026>
- ▶ You are required to have read and consult regularly this web page and pages linked to it
- ▶ “I did not see . . . on the web site” is never a valid excuse
- ▶ Follow instructions to sign up for
 - ▶ Gradescope
 - ▶ Ed Discussions

Grading

- ▶ Course grade
 - ▶ Homework: 0%
 - ▶ Weekly quizzes based on homework: 20%
 - ▶ Midterm: 30%
 - ▶ Final: 50%
 - ▶ Tweaks
- ▶ Overall grading policy
 - ▶ Partial credit for correct answer
 - ▶ Full credit if correct answer is correctly justified
 - ▶ Incorrect logic and calculations will be heavily penalized

Office Hours, Homework, Announcements

- ▶ Office hours
 - ▶ Thursdays 3-5pm, 2 Metrotech 864
- ▶ Announcements
 - ▶ Ed Discussion
 - ▶ Turn on email or text notifications

Overall Approach

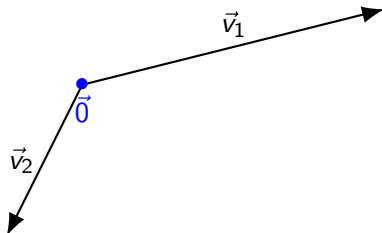
- ▶ Linear algebra is a fundamental language and tool for
 - ▶ Geometry
 - ▶ Physics
 - ▶ Analysis
 - ▶ Any other field where linear functions are used
- ▶ Definitions and facts in linear algebra should not depend on units of length and distance

Vectors in the Plane

- ▶ There is a zero vector, also called the origin

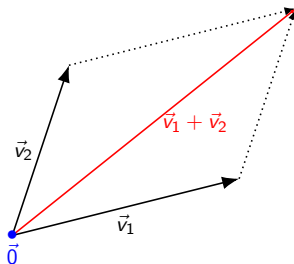
$$\vec{0}$$

- ▶ A vector is an arrow that starts at the origin and has a direction and magnitude (length)

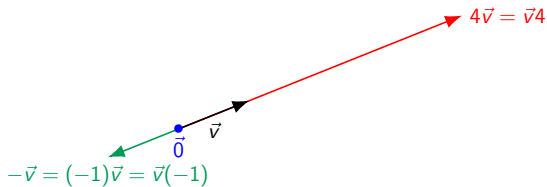


Vector Operations

► Vector addition



► Scalar multiplication



Properties of Vector Addition

- ▶ Notation

$$\begin{aligned} V \times V &\rightarrow V \\ (\vec{v}_1, \vec{v}_2) &\mapsto \vec{v}_1 + \vec{v}_2, \end{aligned}$$

- ▶ Associativity

$$(\vec{v}_1 + \vec{v}_2) + \vec{v}_3 = \vec{v}_1 + (\vec{v}_2 + \vec{v}_3)$$

- ▶ Commutativity

$$\vec{v}_1 + \vec{v}_2 = \vec{v}_2 + \vec{v}_1$$

- ▶ Identity element: There exists an element $\vec{0} \in V$ such that, for any $\vec{v} \in V$,

$$\vec{v} + \vec{0} = \vec{v}$$

- ▶ Inverse element: For each $\vec{v} \in V$, there exists an element, written as $-\vec{v}$, such that

$$\vec{v} + (-\vec{v}) = \vec{0}$$

Properties of Scalar Multiplication

- ▶ Notation

$$\begin{aligned}\mathbb{R} \times V &\rightarrow V \\ (r, \vec{v}) &\mapsto r\vec{v} = \vec{v}r\end{aligned}$$

- ▶ Associativity

$$(r_1 r_2) \vec{v} = r_1 (r_2 \vec{v})$$

- ▶ Distributivity

$$\begin{aligned}(r_1 + r_2) \vec{v} &= r_1 \vec{v} + r_2 \vec{v} \\ r(\vec{v}_1 + \vec{v}_2) &= r\vec{v}_1 + r\vec{v}_2\end{aligned}$$

- ▶ Identity element

$$1\vec{v} = \vec{v}$$

- ▶ Consequences

$$\begin{aligned}0\vec{v} &= \vec{0} \\ (-1)\vec{v} &= -\vec{v}\end{aligned}$$

Another Fundamental Example: $\mathbb{R}^n$

- ▶ $\mathbb{R}^n = \{(v^1, \dots, v^n) : v^1, \dots, v^n \in \mathbb{R}\}$
- ▶ Fundamental operations
 - ▶ Vector addition
 - ▶ Scalar multiplication
- ▶ Same properties of vector addition and scalar multiplication as above
- ▶ The following operations are omitted for now:
 - ▶ Dot product
 - ▶ Cross product

Abstract Vector Space

- ▶ Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$; an element of $\mathbb{F}$ is called a **scalar**
- ▶ An abstract vector space is a set of elements called vectors that have the same fundamental operations and properties as vectors in the plane
- ▶ A **vector space** over $\mathbb{F}$ is a set V with the following:
 - ▶ A special element, denoted $0_V \in V$, called the **zero vector**
 - ▶ An operation called vector addition:

$$\begin{aligned} V \times V &\rightarrow V \\ (v_1, v_2) &\mapsto v_1 + v_2 \end{aligned}$$

- ▶ An operation called scalar multiplication:

$$\begin{aligned} V \times \mathbb{F} &\rightarrow V \\ (v, r) &\mapsto rv = vr \end{aligned}$$

- ▶ Observe that a vector space is always nonempty, because it always contains the zero vector

Properties of Vector Addition

- ▶ Notation

$$\begin{aligned}V \times V &\rightarrow V \\(v_1, v_2) &\mapsto v_1 + v_2,\end{aligned}$$

- ▶ Associativity

$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$$

- ▶ Commutativity

$$v_1 + v_2 = v_2 + v_1$$

- ▶ Identity element: There exists an element $0_V \in V$ such that, for any $v \in V$,

$$v + 0_V = v$$

- ▶ Inverse element: For each $v \in V$, there exists an element, written as $-v$, such that

$$v + (-v) = 0_V$$

Properties of Scalar Multiplication

- ▶ Notation

$$\mathbb{F} \times V \rightarrow V$$
$$(r, v) \mapsto rv = vr$$

- ▶ Associativity

$$(r_1 r_2)v = r_1(r_2 v)$$

- ▶ Distributivity

$$(r_1 + r_2)v = r_1 v + r_2 v$$
$$r(v_1 + v_2) = rv_1 + rv_2$$

- ▶ Identity element

$$1v = v$$

- ▶ Consequences

$$0v = 0$$
$$(-1)v = -v$$

Examples

- ▶ The original example is $\mathbb{F}^n$, whose elements are of the form

$$v = \langle r^1, \dots, r^n \rangle = \begin{bmatrix} r^1 \\ \vdots \\ r^n \end{bmatrix},$$

where $r^1, \dots, r^n \in \mathbb{F}$

- ▶ The set of all solutions to

$$3x - y + 2z = 0 \text{ and } x + 2y - z = 0$$

- ▶ Polynomials of degree 10 or less
- ▶ Polynomials of any degree
- ▶ Continuous functions $f : \mathbb{F} \rightarrow \mathbb{F}$

Why Abstract Vector Spaces Are Useful

- ▶ If we can prove something using only the properties of an abstract vector space, then it has to be true for any specific vector space
- ▶ Often, the statement and proof of a theorem are *easier* for an abstract space than for a specific example such as $\mathbb{R}^2$

Mathematical Grammar

- ▶ Invalid expressions
 - ▶ $a + b$, where $a \in \mathbb{R}$ and $b \in V$
 - ▶ ab , where $a, b \in V$
- ▶ When you write a formula or do a calculation,
 - ▶ Make sure you are adding or multiplying correctly
 - ▶ This is a good way to catch your mistakes

Linear Combination of Vectors

- ▶ Given a nonempty finite set of vectors $v_1, \dots, v_m \in V$ and scalars $r^1, \dots, r^m \in \mathbb{F}$, the vector

$$r^1 v_1 + \dots + r^m v_m$$

is called a **linear combination** of $v_1, \dots, v_m$

- ▶ A linear combination of the empty set is equal to 0
- ▶ Given a subset $S \subset V$, not necessarily finite, the **span** of S is the set of all possible linear combinations of vectors in S

$$\text{span}(S) =$$

$$\{r^1 v_1 + \dots + r^m v_m : m \geq 0, r^1, \dots, r^m \in \mathbb{F} \text{ and } v_1, \dots, v_m \in S\}$$

- ▶ The span of the empty set is $\{0\}$
- ▶ A vector space V is called **finite dimensional** if there is a finite set S of vectors such that

$$\text{span}(S) = V$$

Linearly Independent Set of Vectors

- ▶ A subset $S \subset V$ is **linearly independent** if for any finite subset $\{v_1, \dots, v_r\} \subset S$, any of the following equivalent statements holds:

- ▶ If $a^1, \dots, a^r \in \mathbb{F}$ are scalars such that

$$v_1 a^1 + \dots + v_r a^r = 0,$$

then $a^1 = \dots = a^r = 0$

- ▶ If $a^1, \dots, a^r, b^1, \dots, b^r \in \mathbb{F}$ are scalars such that

$$v_1 a^1 + \dots + v_r a^r = v_1 b^1 + \dots + v_r b^r,$$

then $a^1 = b^1, \dots, a^r = b^r$

- ▶ The empty set is linearly independent

Linearly Dependent Set of Vectors

- ▶ A subset of vectors, $\{v_1, \dots, v_r\}$, is **linearly dependent** if any of the following holds:
 - ▶ The set is not linearly independent
 - ▶ There exist scalars $a^1, \dots, a^r$, where at least one is nonzero, such that

$$v_1 a^1 + \dots + v_r a^r = 0$$

- ▶ The empty set is not linearly dependent, but the set $\{0\}$ is

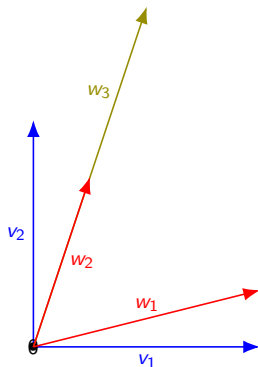
Examples of Linearly Independent and Linearly Dependent Sets

- ▶ $\{(1, 0, 0), (0, 1, 0)\} \subset \mathbb{F}^3$ is linearly independent
- ▶ $\{(1, 0, 0), (0, 1, 0), (1, 1, 0)\} \subset \mathbb{F}^3$ is linearly dependent
- ▶ The empty set is linearly independent
- ▶ $\{0\}$ is linearly dependent

Basis of a Vector Space

- ▶ A subset $S \subset V$ is a **basis** of V if it is a linearly independent set that spans V
- ▶ The only basis of the vector space $\{0\}$ is the empty set

Examples of Bases



- ▶ $\{v_1, v_2\}$ is a basis
- ▶ $\{w_1, w_2\}$ is a basis
- ▶ $\{w_1, w_3\}$ is a basis
- ▶ $\{w_2, w_3\}$ is NOT a basis

Examples of Bases

- Basis of the plane

$$S = \{(1, 0), (0, 1)\}$$

$$T = \{(2, 1), (-1, 1)\}$$

- Basis of $\mathbb{R}^3$

$$U = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

$$V = \{(2, 1, 0), (-1, 1, 0), (0, 1, 1)\}$$

- Not a basis of $\mathbb{R}^3$

$$W = \{(1, 0, 0), (0, 1, 0)\}$$

$$X = \{(2, 1, 0), (-1, 1, 1), (1, 2, 1)\}$$

Notation

- ▶ If $T \subset S$, then

$$S \setminus T = \{s : s \in S \text{ and } s \notin T\}$$

- ▶ $S \setminus T$ is called the complement of T in S

Linear Dependence Lemma (Axler, 2.21)

- ▶ If V is a vector space and $S = \{v_1, \dots, v_k\} \subset V$ is linearly dependent, then there exists an element $v_j \in S$ such that

$$\text{span}(S \setminus \{v_j\}) = \text{span}(S)$$

- ▶ Proof

- ▶ By the definition of linear dependence, there are scalars $s^1, \dots, s^k$, not all zero, such that

$$s^1 v_1 + \dots + s^k v_k = 0$$

- ▶ One of the coefficients is nonzero and we can choose the order of the vectors such that $s^k \neq 0$
- ▶ Therefore,

$$v_k = \frac{-s^1}{s^k} v_1 + \dots + \frac{-s^{k-1}}{s^k} v_{k-1}$$

- ▶ It follows that

$$v_k \in \text{span}(S \setminus \{v_k\})$$

and therefore

$$\text{span}(S \setminus \{v_k\}) = \text{span}(S)$$

Every Finite Dimensional Vector Space Has a Basis

- ▶ Let V be a finite dimensional vector space
- ▶ Suppose $S_m = \{v_1, \dots, v_m\} \subset V$ spans V
- ▶ If S_m is linearly independent, then S_m is a basis
- ▶ If not, then S_m is linearly dependent and, by the lemma above, there is a vector, call it $v_m \in S_m$ such that

$$S_{m-1} = S_m \setminus \{v_m\} = \{v_1, \dots, v_{m-1}\}$$

spans V

- ▶ If S_{m-1} is linearly independent, it is a basis
- ▶ If not, repeat
- ▶ After repeating the steps above a finite number of times, the set S_k is linearly independent

Span of a Linearly Independent Set (Part 1)

- ▶ If $S = \{s_1, \dots, s_m\}$ and $T = \{t_1, \dots, t_n\}$ are linearly independent sets in a vector space V such that

$$\text{span}(S) \subset \text{span}(T),$$

then $m \leq n$

- ▶ Proof

- ▶ If $S = \emptyset$, then $m = 0 \leq n$
- ▶ Since $s_1 \in \text{span}(T)$, there exist scalars $a^1, \dots, a^n$, not all zero, such that

$$s_1 = a^1 t_1 + \dots + a^n t_n$$

- ▶ At least one of the coefficients is nonzero, say a^1 , and therefore

$$t_1 = \frac{1}{a^1} s_1 + \frac{-a^2}{a^1} t_2 + \dots + \frac{-a^n}{a^1} t_n$$

- ▶ It follows that the set $T_1 = \{s_1, t_2, \dots, t_n\}$ is linearly independent and $\text{span}(T_1) = \text{span}(T)$

Span of a Linearly Independent Set (Part 2)

- ▶ Suppose the set $T_j = \{s_1, \dots, s_j, t_{j+1}, \dots, t_n\}$ is linearly independent and $\text{span}(T_j) = \text{span}(T)$
 - ▶ We just proved this for $j = 1$
- ▶ If $j < m$, then since $s_{j+1} \in \text{span}(T)$, there are coefficients $a^1, \dots, a^n$, not all zero (and not the same as the ones before), such that

$$s_{j+1} = a^1 s_1 + \dots + a^j s_j + a^{j+1} t_{j+1} + \dots + a^n t_n$$

- ▶ At least one of the $a^{j+1}, \dots, a^n$ is nonzero, say a^{j+1}
- ▶ From this it follows that

$$t_{j+1} = \frac{1}{a^{j+1}} s_{j+1} + \frac{-a^1}{a^{j+1}} s_1 + \dots + \frac{-a^j}{a^{j+1}} s_j + \frac{-a^{j+2}}{a^{j+1}} t_{j+2} + \dots + \frac{-a^n}{a^{j+1}} t_n$$

- ▶ This implies that the set $T_{j+1} = \{s_1, \dots, s_{j+1}, t_{j+2}, \dots, t_n\}$ is linearly independent and $\text{span}(T_{j+1}) = \text{span}(T)$

Span of a Linearly Independent Set (Part 3)

- ▶ If $m > n$, then, after repeating this step n times, we find that

$$T_n = \{s_1, \dots, s_n\} \subsetneq S$$

is linearly independent and $\text{span}(T_n) = \text{span}(T)$

- ▶ This implies that

$$\text{span}(T) = \text{span}(T_n) \subsetneq \text{span}(S) \subset \text{span}(T),$$

which is a contradiction

- ▶ Therefore, $m \leq n$
- ▶ It also follows that

$$m = n \iff \text{span}(S) = \text{span}(T)$$

Dimension of a Finite Dimensional Vector Space

- ▶ Corollary: Any two bases of V have the same number of elements
 - ▶ If S and T are bases of V , then $\text{span}(S) = \text{span}(T)$ and therefore $m = n$
- ▶ The dimension of a vector space is defined to be the number of elements in a basis
- ▶ A nonempty subset T of a vector space V is a **subspace** of V if for any $p, q \in \mathbb{F}$ and $a, b \in T$,

$$pa + qb \in T$$

- ▶ Every subspace T of V is itself a vector space using the same vector addition and scalar multiplication of V
- ▶ $\dim(T) \leq \dim(V)$ with equality holding if and only if $T = V$