MA-GY7033 Linear Algebra I

Homework 4

Problem 1. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear map given by

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 1 & 2 \end{bmatrix}.$$

(a) Find $\ker T$ and give a basis for it.

(b) Find $\operatorname{im} T$ and give a basis for it.

(c) Compute the nullity and rank of T, and verify the rank-nullity theorem.

(d) Determine whether T is injective and whether it is surjective. Justify both answers.

Problem 2. Answer each question using the rank-nullity theorem.

(a) Suppose $L : V \to W$, where $\dim V = 7$, $\dim W = 5$, and $\dim \ker L = 3$. Find $\operatorname{rank} L$. Is L injective? Is it surjective?

(b) Prove that every injective linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ is an isomorphism.

(c) Prove that no linear map from $\mathbb{R}^6$ to $\mathbb{R}^4$ can be injective.

(d) Prove that no linear map from $\mathbb{R}^3$ to $\mathbb{R}^5$ can be surjective.

Problem 3. Define the subspaces

$$K = \operatorname{span}\{(1,1,0), (0,1,1)\} \subseteq \mathbb{R}^3$$

and

$$S = \operatorname{span}\{(1,2,1)\} \subseteq \mathbb{R}^3.$$

Construct a linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\ker L = K \qquad \text{and} \qquad \operatorname{im} L = S.$$

(a) Give an explicit formula for $L(x, y, z)$.

(b) Write L as a matrix

(c) Prove that its kernel and image are exactly K and S, rather than merely containing these subspaces.

(d) Verify rank-nullity for your map.

Problem 4. Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$L(x, y, z) = (x + y + z,\ x + 2y + 3z).$$

(a) Find a basis for $\ker L$.

(b) Extend this basis to an ordered basis $E = (e_1, e_2, e_3)$ of $\mathbb{R}^3$, with e_1 spanning $\ker L$.

(c) Choose an ordered basis $F = (f_1, f_2)$ of $\mathbb{R}^2$ such that

$$L(e_2) = f_1, \qquad L(e_3) = f_2.$$

(d) Find the matrix M of L with respect to E and F.

(e) Verify directly the identity

$$L(E) = FM.$$

Problem 5. Let $\mathcal{P}_3(\mathbb{R})$ be the vector space of real polynomials of degree at most 3, including the zero polynomial, and define $\mathcal{P}_2(\mathbb{R})$ similarly. Consider the differentiation map

$$D : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R}), \qquad D(p) = p'.$$

(a) Find $\ker D$, $\operatorname{im} D$, the nullity of D, and the rank of D.

(b) Determine whether D is injective and whether it is surjective.

(c) Find the matrix of D with respect to

$$E = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 & x & x^2 \end{bmatrix}.$$

(d) Find a new ordered basis E' of $\mathcal{P}_3(\mathbb{R})$, beginning with a basis of $\ker D$, for which the matrix relative to E' and F is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$