Notes on Huybrechts Complex Geometry
Chapter 5: Applications of Cohomology

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1 Hirzebruch-Riemann-Roch Theorem

1.1 Statement and Examples

Recall the Riemann-Roch Formula

$$\chi(C, E) = \deg(E) + \operatorname{rank}(E) \cdot (1 - g(C)) \tag{1.1}$$

for a holomorphic vector bundle E and complex curve C, as well as for line bundles on surfaces. Here the LHS is the **Euler-Poincaré characteristic**

$$\chi(X, E) := \sum_{i=0}^{\dim(X)} (-1)^i h^i(X, E),$$

for a holomorphic vector bundle E on a compact complex manifold X.

The Hirzebruch-Riemann-Roch (HRR) formula generalizes this. The proof of HRR follows from further generalizations, including the Grothendieck-Riemann-Roch formula and Atiyah-Singer index theorem.

Question 1.1. What is the degree of a vector bundle?

See p.88-89 for holomorphic line bundle, assuming X is projective. See ex.4.4.1 for the relation to Chern class.

For a connected compact curve C, one has $H^2(C, \mathbb{Z}) \cong H_0(C, \mathbb{Z}) = \mathbb{Z}$ by Poincaré Duality using $\int_C$. Then for a complex line bundle L on C, its first Chern Class $c_1(L) \in H^2(C, \mathbb{Z}) = \mathbb{Z}$ is defined as the degree of L. For complex vector bundle E of higher rank, we define $\deg(E) := \deg(\det E)$.

Theorem 1.2 (Hirzebruch-Riemann-Roch)**.** *Let E be a holomorphic vector bundle on a compact complex manifold X. Then its Euler-Poincaré characteristic equals*

$$\chi(X, E) = \int_X \operatorname{ch}(E) \operatorname{td}(X).$$

Indeed only the $H^{2n}(X, \mathbb{C})$ component of $\operatorname{ch}(E)\operatorname{td}(X)$ contributes to the integral, where $n = \dim_{\mathbb{C}}(X)$. This is

$$[\operatorname{ch}(E)\operatorname{td}(X)]_{2n} = \sum_i \operatorname{ch}_i(E)\operatorname{td}_{n-i}(X).$$

The formula agrees with the additivity of the Chern character. Recall that given any SES of holomorphic vector bundles

$$0 \to E_1 \to E \to E_2 \to 0$$

one has

$$\chi(X, E) = \chi(X, E_1) + \chi(X, E_2).$$

Indeed we already know that

$$\operatorname{ch}(E) = \operatorname{ch}(E_1 \oplus E_2) = \operatorname{ch}(E_1) + \operatorname{ch}(E_2).$$

As a consequence of HRR, $\operatorname{ch}(E)$ and hence $\chi(X, E)$ depends only on the underlying complex vector bundle E, independent of its holomorphic structure. As a special case, for holomorphic line bundles L_1, L_2 over X, if $c_1(L_1) = c_1(L_2)$, then $\chi(X, L_1) = \chi(X, L_1)$, even if different holomorphic structures give $h^0(X, L_1) \neq h^0(X, L_2)$. Here $H^0(X, L)$ is the space of global **holomorphic** sections. The reason is **the Chern character can be computed from the Chern class (proof using formal Chern roots?)**, and in the line bundle case $\operatorname{ch}(L) = \exp(c_1(L)) = 1 + c_1(L) + \frac{1}{2}c_1(L)^2 + \dots$.

Question 1.3. Does the Chern class uniquely determine the complex vector bundle, or the complex line bundle?

Question 1.4. Can we relate the pullback vector bundle using HRR? The Chern class, Chern character, and Todd class all respect pullback. Given (nice) holomorphic map $f : Y \to X$, can we get

$$\chi(Y, f^*E) = \int_Y f^* \left(\operatorname{ch}(E) \operatorname{td}(X)\right) = \deg(f) \cdot \chi(X, E)?$$

Answer: this is true for generic finitely-sheeted covering maps, where "generic" implies f is covering map except on a set of measure zero, so that it is ignored by the integral.

Example 1.5 (Line bundles on a curve). Let C be a connected compact curve and $L \in \operatorname{Pic}(C)$. Then HRR gives

$$\chi(C, L) = \int_C c_1(L) + \frac{c_1(C)}{2} = \deg(L) + \frac{\deg(\mathcal{K}_C^*)}{2}.$$

In particular, if $L = \mathcal{O}_C$, the curvature on L vanishes so that

$$\chi(C, L) = \frac{\deg(\mathcal{K}_C^*)}{2}.$$

Then

$$\deg(\mathcal{K}_C) = -2\chi(C, L) = 2(h^1(C, L) - h^0(C, L)) = 2(h^1(C, L) - 1)$$

by maximum principle and compactness of C. If we define the genus of C as

$$g(C) := \frac{\deg(\mathcal{K}_C)}{2} + 1, \tag{1.2}$$

then

$$g(C) = h^1(C, \mathcal{O}_C) = h^0(C, \mathcal{K}_C)$$

by Serre duality $h^{0,1} = h^{1,0}$. Therefore,

$$\chi(C, \mathcal{O}_C) = -\frac{\deg(\mathcal{K}_C)}{2} = \deg(\mathcal{O}_C) + \operatorname{rank}(\mathcal{O}_C) \cdot (1 - g(C)),$$

which yields the Riemann-Roch formula (1.1).

Remark 1.6. Note that $H^i(C, \mathcal{O}_C) = H^{0,i}(C) = 0$ for $i > 1$, as $\dim_{\mathbb{C}}(C) = 1$.

Question 1.7. Topological meaning of the genus of a complex curve defined in (1.2)?

We start from the topological equality $H^1(C, \mathbb{C}) = \mathbb{C}^{2g}$ for a complex curve C of genus g. For example, $C = \mathbb{C}/\Lambda$ is torus of genus 1. By Hodge decomposition, $H^{1,0}(C) = H^{0,1}(C) = \mathbb{C}^g$. Since C has complex dimension one, the holomorphic 1-form bundle is exactly the canonical bundle: $\Omega_C = \mathcal{K}_C$. Thus $H^{1,0}(C) = H^0(C, \mathcal{K}_C)$, and $g(C) = h^0(C, \mathcal{K}_C)$. This agrees with the calculation above, and hence the genus is defined more generally as above.

Example 1.8 (Line bundles on a surface). Let X be a compact complex surface. First consider the trivial bundle. In this case HRR gives **Noether's formula**

$$\begin{aligned}\chi(X,\mathcal{O}_X) &= h^0(X,\mathcal{O}_X) - h^1(X,\mathcal{O}_X) + h^2(X,\mathcal{O}_X)\\ &= \int_X \frac{c_1(X)^2 + c_2(X)}{12}\end{aligned}$$

using ex.4.4.5. If L is any line bundle, then

$$\begin{aligned}\chi(X,L) &= h^0(X,L) - h^1(X,L) + h^2(X,L)\\ &= \int_X \operatorname{td}_2(X) + \operatorname{ch}_1(L)\operatorname{td}_1(X) + \operatorname{ch}_2(L)\\ &= \chi(X,\mathcal{O}_X) + \int_X c_1(L)\frac{c_1(X)}{2} + \frac{c_1(L)^2}{2}\\ &= \chi(X,\mathcal{O}_X) + \int_X \frac{c_1(L)(c_1(L)+c_1(X))}{2},\end{aligned}$$

also written as

$$\chi(X,L) = \chi(X,\mathcal{O}_X) + \frac{L.(L-\mathcal{K}_X)}{2},$$

as $c_1(X) = c_1(\mathcal{K}_X^*) = -c_1(\mathcal{K}_X)$.

Example 1.9 (Line bundles on a torus). Let $X := \mathbb{C}^n/\Gamma$ be a complex torus. Then all characteristic classes of X are trivial, as X is **flat** and the curvature on $\mathcal{T}_X$ vanishes. Then for any holomorphic line bundle L over X,

$$\chi(\mathbb{C}^n/\Gamma, L) = \int_{\mathbb{C}^n/\Gamma} \frac{c_1(L)^n}{n!}.$$

In particular, when $n=1$, X is a complex torus of genus $g(X)=1$, so

$$\chi(X,L) = \deg(L) = \deg(L) + \operatorname{rank}(L)\cdot(1-g(X)),$$

which coincides with the Riemann-Roch formula.

Remark 1.10. Following the definition of genus above, when $X=\mathbb{C}/\Gamma$, we have

$$\deg(\mathcal{K}_X) = 0.$$

Indeed

$$\mathcal{K}_X = \Omega_X \cong \mathcal{O}_X$$

under the map $f dz \leftrightarrow f$.

1.2 Application

We now consider an application of HRR. Let X be a compact complex manifold of dimension n. Define the **arithmetic genus** of X as

$$p_a(X) := (-1)^n(\chi(X,\mathcal{O}_X) - 1).$$

Indeed if X is a curve, by example 1.5 above, this agrees with the geometric genus:

$$p_a(X) = 1 - (1 - h^1(X,\mathcal{O}_X)) = g(X).$$

Another geometric quantity we have seen is

$$\sum_{p,q=0}^{n} (-1)^q h^{p,q}(X) = \sum_{p=0}^{n} \chi(X,\Omega_X^p),$$

which equals $\operatorname{sgn}(X)$, the signature of the (non-degenerate) intersection form on the middle cohomology class when $\dim X = n$ is even. Both expressions, $\chi(X, \mathcal{O}_X)$ and $\sum_p \chi(X, \Omega_X^p)$, are evaluations of the Hirzebruch χ_y-genus:

Definition 1.11. The **Hirzebruch χ_y-genus** of a compact complex manifold of dimension n is the polynomial

$$\chi_y := \sum_{p=0}^{n} \chi(X, \Omega_X^p) y^p = \sum_{p,q=0}^{n} (-1)^q h^{p,q}(X) y^p.$$

We can calculate the Hirzebruch χ_y-genus using HRR and Chern roots:

Proposition 1.12. *Let γ_i denote the formal Chern roots of $\mathcal{T}_X$. Then*

$$\chi_y = \int_X \prod_{i=1}^{n} (1 + ye^{-\gamma_i}) \frac{\gamma_i}{1 - e^{-\gamma_i}}.$$

Proof. First, using HRR, we get

$$\begin{aligned} \chi_y &= \int_X \left(\sum_{p=0}^{n} \operatorname{ch}(\Omega_X^p) y^p \right) \operatorname{td}(X) \\ &= \int_X \operatorname{ch} \left(\bigoplus_{p=0}^{n} \Omega_X^p y^p \right) \operatorname{td}(X). \end{aligned}$$

Now compute each term separately. By definition of Todd class,

$$\operatorname{td}(X) = \prod_{i=1}^{n} \frac{\gamma_i}{1 - e^{-\gamma_i}}$$

under the diagonalization $(\gamma_1, ..., \gamma_n)$ of $iF_\nabla / 2\pi$. We claim that

$$\operatorname{ch} \left(\bigoplus_{p=0}^{n} \Omega_X^p y^p \right) = \prod_{i=1}^{n} (1 + ye^{-\gamma_i}),$$

which would complete the proof. This follows from the more general formula for the Chern character of the exterior algebra.

□

Question 1.13. Chern character of the exterior algebra?

How does the diagonalization of $iF_\nabla/2\pi$ relate to the decomposition of the vector bundle into line bundles? Hard question; save for later.

Let V be a holomorphic vector bundle of rank n. Since Chern character is local identity, we apply the local splitting principle and **assume wlog**

$$V = \bigoplus_i L_i$$

for line bundles L_i. Then the exterior algebra

$$\bigwedge V := \bigoplus_{p=0}^{n} \wedge^p(V)$$

decomposes as

$$\bigwedge V = \bigotimes_{i=1}^{n} \bigwedge L_i.$$

Hence the Chern character decomposes

$$\mathrm{ch}(\bigwedge V) = \prod_{i=1}^{n} \mathrm{ch}(\bigwedge L_i).$$

In our case $V = \Omega_X y$. The dual $\mathcal{T}_X = \Omega_X^*$ decomposes as a direct sum of line bundles whose curvature $iF_\nabla/2\pi$ is given by the Chern roots γ_i. Thus Ω_X decomposes as sum of line bundles L_i with curvature $-\gamma_i$. The Chern character for $\wedge L_i y$ is thus

$$\mathrm{ch}(\bigwedge L_i y) = \mathrm{ch}(\mathcal{O}_X \oplus L_i y) = 1 + y\,\mathrm{ch}(L_i) = 1 + ye^{-\gamma_i}.$$

Therefore,

$$\mathrm{ch}(\bigwedge V) = \mathrm{ch}\left(\bigoplus_{p=0}^{n} \Omega_X^p y^p\right) = \prod_{i=1}^{n}(1 + ye^{-\gamma_i}).$$

Some important special values of the Hirzebruch χ_y-genus:

1. $y = 0$: $\chi_{y=0} = \chi(X, \mathcal{O}_X) = \int_X \mathrm{td}(X)$ gives the arithmetic genus.
2. $y = 1$: $\chi_{y=1} = \sum_{p,q=0}^{n}(-1)^q h^{p,q}(X) = \mathrm{sgn}(X)$ if X is compact Kähler of even dimension n, by Cor.3.3.18 and complex conjugation $h^{p,q} = h^{q,p}$. Combining with the proposition above, we get **Hirzebruch signature theorem** for compact Kähler manifolds of even dimension:

$$\mathrm{sgn}(X) = \chi\left(\bigoplus_{p=0}^{n} \Omega_X^p\right) = \int_X \mathrm{ch}\left(\bigoplus_{p=0}^{n} \Omega_X^p\right) \mathrm{td}(X) = \int_X L(X),$$

   where $L(X)$ is the **L-genus**. By proposition above, $L(X)$ is given in terms of the Chern roots by

$$L(X) = \prod_{i=1}^{n} \gamma_i \frac{(1 + e^{-\gamma_i})}{1 - e^{-\gamma_i}} = \prod_{i=1}^{n} \gamma_i \cdot \coth\left(\frac{\gamma_i}{2}\right).$$

   The same result holds for any compact complex manifold of even dimension.

   Question 1.14. Proof for the non-Kähler case? Essentially it suffices to prove the first equality:

$$\mathrm{sgn}(X) = \chi\left(\bigwedge \Omega_X\right) \equiv \int_X L(X).$$

   This follows from the general Atiyah-Singer index theorem.

   Is there another defnition of the L-genus without using Chern roots? L-genus is usually defined this way.

3. $y = -1$: Suppose X is compact Kähler manifold of dimension n. Then

$$\chi_{y=-1} = \sum_{p,q=0}^{n} (-1)^{p+q} h^{p,q}(X) = \sum_{i=0}^{2n} (-1)^i b_i(X) = e(X)$$

   is the Euler characteristic of X, where b_i are the Betti numbers. By proposition above and definition of Chern class,

$$e(X) = \int_X \prod_{i=1}^{n} \gamma_i = \int_X c_n(X).$$

   The equality

$$e(x) = \int_X c_n(X)$$

is the **Gauss-Bonnet formula**, which holds more generally for compact complex manifolds of dimension n. $\int_X c_n(X)$ also generalizes to **Chern numbers** for compact complex manifolds, which are of form

$$\int_X c_{i_1}(X)\dots c_{i_k}(X)$$

such that $i_1 + \cdots + i_k = \dim(X)$.

1.3 Generalizations

Theorem 1.15 (Grothendieck-Riemann-Roch formula)**.** *Let $f : X \to Y$ be a smooth projective morphism of smooth projective varieties. Then for any coherent sheaf $\mathcal{F}$ (e.g. a vector bundle) on X one has*

$$\operatorname{ch}\left(\sum_i (-1)^i R^i f_* \mathcal{F}\right) \operatorname{td}(Y) = f_* \left(\operatorname{ch}(\mathcal{F}) \operatorname{td}(X)\right)$$

in the rational Chow group $\mathrm{CH}(Y)_{\mathbb{Q}}$ or in $H^*(Y, R)$.

HRR is a special case of GRR above. Consider $f : X \to \{\mathrm{pt}\}$ with a vector bundle $\mathcal{F} = E$ on X. Then on RHS, $f_* = \int_X$. On LHS, $\operatorname{td}(\{\mathrm{pt}\}) = 1$, and $R^i f_* E = H^i(X, E)$ as a vector bundle, or simply a vector space, over $\{\mathrm{pt}\}$. Such a vector bundle is trivially flat, so its Chern character is the dimension of $H^*(X, E)$. Therefore, the LHS of GRR formula is $\sum_i (-1)^i h^i(X, E) = \chi(X, E)$.

Theorem 1.16 (Atiyah-Singer index theorem)**.** *Let $D : \Gamma(E) \to \Gamma(F)$ be an elliptic differential operator between vector bundles E and F on a compact oriented differentiable manifold M. Then the analytic index of D equals the topological index of D. Here the analytic index is defined as* $\dim\ker D - \dim\operatorname{coker} D$.

Question 1.17. How is topological index defined in terms of the characteristic classes of E and F?
To deduce HRR from above, consider $D = \Delta_{\overline{\partial}_E}$ with $F = E$?

Note that **locally free coherent sheaves are exactly holomorphic vector bundles**. HRR formula computes the Euler-Poincaré characteristic of vector bundles. What about coherent sheaves that are not locally free, e.g. the ideal sheaf $\mathcal{I}_Z$ of a submanifold $Z \subset X$? We can utilize the following result:

Proposition 1.18. *Any coherent sheaf $\mathcal{F}$ on a projective manifold has a finite resolution*

$$0 \to E_n \to \cdots \to E_1 \to \mathcal{F} \to 0,$$

where E_i are locally free coherent sheaves, i.e. (holomorphic) vector bundles.

HRR formula computes $\chi(X, E_i)$, and for an exact sequence the Euler-Poincaré characteristic satisfies

$$\chi(X, \mathcal{F}) = \sum_{i=1}^{n} (-1)^{i-1} \chi(X, E_i).$$

See Cor.B.0.37 for the special case of a short exact sequence. Combining, we get

$$\chi(X, \mathcal{F}) = \int_X \sum_{i=1}^{n} (-1)^{i-1} \operatorname{ch}(E_i) \operatorname{td}(X) \stackrel{?}{=} \int_X \operatorname{ch}(\mathcal{F}) \operatorname{td}(X)$$

Question 1.19. Can we define the Chern character of a coherent sheaf in this way using the locally free resolution? How is $\operatorname{ch}(\mathcal{F})$ defined as in the statement of GRR? See ex.4.4.11 for the approach using Atiyah class, but how to define Atiyah class for sheaves using Čech cocycle? Hard question. It is easier to consider the first Chern class of a coherent sheaf.

The ideal sheaf is not locally free, because at a point $z \in Z \subset X$, any section takes only one value 0, so that locally near z the sheaf cannot be trivialized as $U \times \mathbb{C}^r$ for any $r > 0$. However, the line bundle $\mathcal{O}(-Y)$ is locally free, and by Lem.2.3.22, the image of $\mathcal{O}(-Y) \to \mathcal{O}_X$ is $\mathcal{I}_Y$?

As another example, consider the Euler-Poincaré characteristic of the structure sheaf $\mathcal{O}_Y$ of a smooth hypersurface $Y \subset X$. Recall the short exact sequence of sheaves

$$0 \to \mathcal{O}(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0,$$

which is a locally free resolution of $\mathcal{O}_Y$. Then HRR gives

$$\chi(Y, \mathcal{O}_Y) = \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{O}(-Y)) = \int_X \left(1 - e^{-[Y]}\right) \mathrm{td}(X),$$

where we use that $c_1(O(-Y)) = -[Y]$ by Prop.4.4.13. Thus we can consider $\mathrm{ch}(\mathcal{O}_Y)$ as $1 - e^{-[Y]}$, **but** the second term is $\mathrm{td}(X)$ instead of $\mathrm{td}(Y)$?

Question 1.20. The relation Cor.B.0.37 holds for short exact sequence of sheaves over different base manifolds? This is true only in the special case above, where $Y \subset X$ is smooth hypersurface. In this case, $H^*(X, i_*\mathcal{F}) \cong H^*(Y, \mathcal{F})$, where i_* is the pushforward by inclusion $Y \hookrightarrow X$.

See final paragraph of §5.1. HRR, GRR and AS all hold for arbitrary complex manifolds and coherent sheaves?

2 Kodaira Vanishing Theorem

Kodaira Vanishing Theorem, combined with Hirzebruch-Riemann-Roch formula, yields bounds for the dimension of $H^0(X, L)$, the space of global holomorphic sections of a line bundle L over a compact Kähler manifold X. Below L is always **holomorphic** line bundle, and E **holomorphic** vector bundle.

2.1 Statement and Proof of the Theorem

Definition 2.1. Let X be a complex manifold. A line bundle L on X is called a **positive line bundle** if its first Chern class $c_1(L) \in H^2(X, \mathbb{R})$ can be represented by a closed positive real (1, 1)-form.

Recall that a real (1, 1)-form α is called positive if for all holomorphic tangent vectors $0 \neq v \in T^{1,0}X$ one has

$$-i\alpha(v, \overline{v}) > 0.$$

Semipositivity is defined analogously with the $\geq$ sign.

We first list some easy observations.

- Writing a real (1, 1)-form locally as

$$\alpha = \frac{i}{2} \sum_{i,j=1}^{n} h_{ij} dz_i \wedge d\overline{z}_j,$$

  where (h_{ij}) is hermitian matrix, the (semi-)positivity of α is equivalent to the (semi-)positive-definiteness of (h_{ij}).

- Recall that we can recover the hermitian structure on X from the fundamental form and the almost complex structure induced by X. Therefore, a complex manifold admitting a positive line bundle is automatically Kähler. See Lem.3.1.7.

- By ex.4.4.10, if X is compact Kähler, then each closed real (1, 1)-form representing $c_1(L)$ is the curvature of the Chern connection for some Hermitian structure on L. Therefore, a holomorphic line bundle L on X is positive if and only if there exists a Hermitian structure on L whose induced curvature is positive as a real (1, 1)-form. We always suppress the factor $i/2\pi$ for curvatures.

Theorem 2.2 (Kodaira Vanishing Theorem)**.** *Let L be a holomorphic positive line bundle on compact Kähler manifold X of dimension n. Then*

$$H^{p,q}(X,L) = H^q(X,\Omega_X^p \otimes L) = 0, \quad \text{for } p+q>n.$$

We start from a few preliminary lemmas to prove the Kodaira vanishing theorem.

Let (E,h) be a holomorphic vector bundle over X with fixed Hermitian structure. Recall from §4.1 two operators $\overline{\partial}_E$ and $\overline{\partial}_E^*$ on $\mathcal{A}^{p,q}(E)$. We also extend the Lefschetz operator L and dual Lefschetz operator Λ to $\mathcal{A}^{p,q}(E)$ by acting on the (p,q)-form part, i.e. $L = L\otimes 1$, $\Lambda = \Lambda \otimes 1$. Since

$$[\Lambda, L] = (n-(p+q))\cdot \mathrm{Id}$$

on $\mathcal{A}^{p,q}(X)$ by Prop.1.2.26, the same relation holds after extension to E. Another Kähler identity extended to E is

Lemma 2.3 (Nakano Identity)**.** *Let ∇ be the Chern connection on (E,h). Then*

$$[\Lambda, \overline{\partial}_E] = -i\left(\nabla_E^{1,0}\right)^*,$$

where

$$\left(\nabla_E^{1,0}\right)^* := -\overline{*}_{E^*}\circ \nabla_{E^*}^{1,0}\circ \overline{*}_E.$$

Compare this with the Kähler identity on $\mathcal{A}^{p,q}(X)$ (Prop.3.1.12):

$$[\Lambda,\overline{\partial}] = -i\partial^*.$$

Indeed Nakano identity reduces to this when $E=\mathcal{O}_X$, in which case $\nabla = d$.

Proof. The statement is local, so we can work under orthonormal trivialization

$$\psi : E|_U \cong U\times \mathbb{C}^r.$$

Then $\overline{*}_E$ equals $\overline{*}$, the complex conjugation of the $\mathbb{C}$-linear Hodge operator. Write the connection on E with respect to ψ as

$$\nabla_E = d + A,$$

where A is $r\times r$ matrix of 1-forms s.t. $A^* = -A$. We have the induced connection on E^*

$$\nabla_{E^*} = d + A^* = d - A^t.$$

Then we compute

$$\overline{\partial}_E = (\nabla_E)^{0,1} = \overline{\partial} + A^{0,1},$$

and

$$\left(\nabla_E^{1,0}\right)^* = -\overline{*}\circ\left(\partial - \left(A^{1,0}\right)^t\right)\circ\overline{*} = -*\circ\overline{\partial}\circ * - \left(A^{1,0}\right)^* = \overline{\partial}^* - \left(A^{1,0}\right)^*.$$

Using Kähler identity from Prop.3.1.12, we have

$$[\Lambda,\overline{\partial}_E] + i\left(\nabla_E^{1,0}\right)^* = [\Lambda,\overline{\partial}] + [\Lambda, A^{0,1}] + i\overline{\partial}^* - i\left(A^{1,0}\right)^* = [\Lambda, A^{0,1}] - i\left(A^{1,0}\right)^*.$$

Thus $[\Lambda,\overline{\partial}_E] + i\left(\nabla_E^{1,0}\right)^*$ is a linear operator (**over complex differentiable functions on X**). We choose the orthonormal trivialization by Rem.4.2.5 such that $A(x_0)=0$, for each fixed point $x_0\in U$. Then $[\Lambda,\overline{\partial}_E] + i\left(\nabla_E^{1,0}\right)^* = 0$ at x_0. This completes the proof.

□

Remark 2.4. Calculate ∇_{E^*}. $\nabla^*(f)(s) = d(f(s)) - f(\nabla(s))$, for any section f of E^*, which can be identified as a map into $(\mathbb{C}^r)^* \cong \mathbb{C}^r$. Note that $f(s) = \langle s, \overline{f}\rangle$ for the inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{C}^r$. Then

$$\nabla^*(f)(s) = d\langle s, \overline{f}\rangle - \langle (d+A)s, \overline{f}\rangle = \langle s, d\overline{f}\rangle - \langle s, \overline{A^t f}\rangle.$$

Therefore, $\nabla^* = d - A^t$. For line bundle indeed $A^t = A$.

The definition

$$\left(\nabla_E^{1,0}\right)^* := -\overline{*}_{E^*} \circ \nabla_{E^*}^{1,0} \circ \overline{*}_E$$

makes sense as the formal adjoint of $\nabla_E^{1,0}$. To see this, as in Lem.4.1.12, we compute for $\alpha \in \mathcal{A}^{p,q}(X,E), \beta \in \mathcal{A}^{p+1,q}(X,E)$,

$$\begin{aligned}
\left(\alpha, -\overline{*}_{E^*} \circ \nabla_{E^*}^{1,0} \circ \overline{*}_E\beta\right) &= -\int_X \alpha \wedge \overline{*}_E\overline{*}_{E^*}\nabla_{E^*}^{1,0}\overline{*}_E\beta \\
&= (-1)^{p+q-1}\int_X \alpha \wedge \nabla_{E^*}^{1,0}\overline{*}_E\beta \\
&= (-1)^{p+q-1}\int_X \alpha \wedge \left(\partial - (A^{1,0})^t\right)\overline{*}_E\beta \\
&= \int_X (\partial + A^{1,0})\alpha \wedge \overline{*}_E\beta \\
&= \left(\nabla_E^{1,0}\alpha, \beta\right).
\end{aligned}$$

Another useful lemma.

Lemma 2.5. *Let (E,h) be a Hermitian holomorphic vector bundle over compact Kähler manifold (X,g). Let ∇ be the Chern connection on E, and $\alpha \in \mathcal{H}^{p,q}(X,E)$ any harmonic form. Then*

$$\frac{i}{2\pi}\left(F_\nabla\Lambda(\alpha), \alpha\right) \le 0,$$

and

$$\frac{i}{2\pi}\left(\Lambda F_\nabla(\alpha), \alpha\right) \le 0.$$

Proof. Since the Chern connection satisfies $\nabla^{0,1} = \overline{\partial}_E$, and $F_\nabla \in \mathcal{A}^{1,1}(X, \mathrm{End}(E))$, we get

$$F_\nabla = \nabla^{1,0} \circ \overline{\partial}_E + \overline{\partial}_E \circ \nabla^{1,0}.$$

α is harmonic, meaning

$$\overline{\partial}_E\alpha = \overline{\partial}_E^*\alpha = 0.$$

Then we compute

$$\begin{aligned}
i\left(F_\nabla\Lambda(\alpha), \alpha\right) &= i\left(\nabla^{1,0}\overline{\partial}_E\Lambda(\alpha), \alpha\right) + i\left(\overline{\partial}_E\nabla^{1,0}\Lambda(\alpha), \alpha\right) \\
&= i\left(\overline{\partial}_E\Lambda(\alpha), \left(\nabla^{1,0}\right)^*\alpha\right) + i\left(\nabla^{1,0}\Lambda(\alpha), \overline{\partial}_E^*\alpha\right) \\
&= \left(\overline{\partial}_E\Lambda(\alpha), -i\left(\nabla^{1,0}\right)^*\alpha\right) && \overline{*}_E \text{ is } \mathbb{C}\text{-antilinear} \\
&= \left(\overline{\partial}_E\Lambda(\alpha), [\Lambda, \overline{\partial}_E]\alpha\right) && \text{Nakano Identity} \\
&= -\left(\overline{\partial}_E\Lambda(\alpha), \overline{\partial}_E\Lambda(\alpha)\right) \\
&\le 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
i\left(\Lambda F_{\nabla}(\alpha),\alpha\right) &= i\left(\Lambda\overline{\partial}_E\nabla^{1,0}\alpha,\alpha\right)\\
&= i\left(\overline{\partial}_E\Lambda\nabla^{1,0}\alpha,\alpha\right)+\left(\left(\nabla^{1,0}\right)^*\nabla^{1,0}\alpha,\alpha\right)\\
&= i\left(\Lambda\nabla^{1,0}\alpha,\overline{\partial}_E^*\alpha\right)+\left(\nabla^{1,0}\alpha,\nabla^{1,0}\alpha\right)\\
&= \left(\nabla^{1,0}\alpha,\nabla^{1,0}\alpha\right)\\
&\geq 0.
\end{aligned}$$

□

Proof of Kodaira Vanishing Theorem. Since L is positive line bundle, as discussed above we can choose a Hermitian structure on L whose induced curvature is closed positive real (1, 1)-form, i.e. $(i/2\pi)F_{\nabla}$ is a Kähler form on X. We fix this Kähler strcture. Then the Lefschetz opeartor L is simply the wedge product by $(i/2\pi)F_{\nabla}$. Using the Kähler identity

$$[\Lambda,L]=(n-(p+q))\cdot\mathrm{Id}$$

and the lemma above, we compute for any harmonic form $\alpha\in\mathcal{H}^{p,q}(X,L)$,

$$(n-(p+q))\|\alpha\|^2=([\Lambda,L]\alpha,\alpha)=\frac{i}{2\pi}\left([\Lambda,F_{\nabla}]\alpha,\alpha\right)\geq 0.$$

Therefore, $0=\mathcal{H}^{p,q}(X,L)\cong H^{p,q}(X,L)\cong H^q(X,\Omega_X^p\otimes L)$ by Hodge Theory, for any $p+q>n$.

□

2.2 Applications of Kodaira Vanishing

Example 2.6. Consider $\mathcal{O}(1)$ on $\mathbb{P}^n$. It is a positive line bundle since

$$\frac{i}{2\pi}F_{\nabla}=\omega_{FS}$$

once we fix the natural hermitian structure induced by the global sections $z_0,\ldots,z_n$ on $\mathbb{P}^n$. See Exa.4.3.12. Recall that

$$c_1(L_1\otimes L_2)=c_1(L_1)+c_1(L_2)$$

for holomorphic line bundles L_1 and L_2. Thus $\mathcal{O}(m)$ is positive line bundle for each $m>0$. By Kodaira Vanishing Theorem,

$$H^q(\mathbb{P}^n,\Omega^p_{\mathbb{P}^n}\otimes\mathcal{O}(m))=0$$

for all $p+q>n$, $m>0$. Prop.2.4.3 shows

$$\Omega^n_{\mathbb{P}^n}=\mathcal{K}_{\mathbb{P}^n}\cong\mathcal{O}(-n-1),$$

so in particular

$$H^q(\mathbb{P}^n,\mathcal{O}(m))=0$$

for all $q>0$, $m\geq -n$. By Serre duality (Prop.4.1.15),

$$H^q(\mathbb{P}^n,\mathcal{O}(m))=\begin{cases}0 & 0<q<n\\ 0 & q=0,m<0\\ 0 & q=n,m>-n-1\end{cases}.$$

For the first case, suppose $m<-n$. Then

$$H^q(\mathbb{P}^n,\mathcal{O}(m))=H^{n,q}(\mathbb{P}^n,\mathcal{O}(m+n+1))\cong H^{0,n-q}(\mathbb{P}^n,\mathcal{O}(-m-n-1))^*=0$$

from above as $n - q > 0$, $-m - n - 1 \geq 0$. For the second case,

$$H^0(\mathbb{P}^n, \mathcal{O}(m)) \cong H^{0,n}(\mathbb{P}^n, \mathcal{O}(-m-n-1))^* = 0$$

as $n > 0$, $-m - n - 1 \geq -n$. The third case is trivial.

We try to gather all information about $H^0(\mathbb{P}^n, \mathcal{O}(m))$. From above,

$$H^n(\mathbb{P}^n, \mathcal{O}(m)) \cong H^0(\mathbb{P}^n, \mathcal{O}(-m-n-1))^*.$$

By Prop.2.4.1,

$$H^0(\mathbb{P}^n, \mathcal{O}(m)) = \mathbb{C}[z_0, ..., z_n]_m,$$

the space of homogeneous polynomials in $\mathbb{C}[z_0, ..., z_n]$ of degree m.

Another consequence of the Kodaira vanishing theorem is the **weak Lefschetz theorem**.

Proposition 2.7. *Let X be a compact Kähler manifold of dimension n, and let $Y \subset X$ be a smooth hypersurface such that the induced line bundle $\mathcal{O}(Y)$ is positive. Then the canonical restriction map*

$$H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$$

is bijective for $k \leq n-2$ and injective for $k \leq n-1$.

Recall that $c_1(\mathcal{O}(Y)) = [Y] \in H^2(X, \mathbb{R})$, so that ex.3.3.5 gives the injectivity part of weak Lefschetz theorem.

Proof. Y inherits Kähler structure from X via the inclusion map. The pullback of forms by inclusion preserves the type. Thus we have bidegree decomposition

$$H^k = \bigoplus_{p+q=k} H^{p,q}$$

for both X and Y, and the restriction map $H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$ is compatible with it. Thus we want to show:

i) $H^q(X, \Omega_X^p) \to H^q(Y, \Omega_Y^p)$ is bijective for $p + q \leq n - 2$,

ii) $H^q(X, \Omega_X^p) \to H^q(Y, \Omega_Y^p)$ is injective for $p + q \leq n - 1$.

We use the structure sequence

$$0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0,$$

and the dual of the normal bundle sequence

$$0 \to \mathcal{N}_{Y/X}^* \to (\Omega_X)\,|_Y \to \Omega_Y \to 0.$$

Y is defined by a section in $H^0(X, \mathcal{O}(Y))$, so by Prop.2.4.7, $\mathcal{N}_{Y/X}^* \cong \mathcal{O}_Y(-Y)$.

Twist the structure sequence by Ω_X^p to get

$$0 \to \Omega_X^p(-Y) \to \Omega_X^p \to \Omega_X^p|_Y \to 0. \tag{2.1}$$

Take the p-th exterior product of the dual of the normal bundle sequence to get (see ex.2.2.2)

$$0 \to \Omega_Y^{p-1}(-Y) \to \Omega_X^p|_Y \to \Omega_Y^p \to 0. \tag{2.2}$$

(Recall from ex.2.2.3 that $(\Omega_X^p)^* \otimes \mathcal{K}_X \cong \Omega_X^{n-p}$). Then by positivity of $\mathcal{O}(Y)$ and Kodaira vanishing theorem,

$$\begin{aligned} H^q(X, \Omega_X^p(-Y)) &= H^{n-p,n-q}(X, \mathcal{O}(Y))^* \\ &= H^{n-q}(X, \Omega_X^{n-p} \otimes \mathcal{O}(Y))^* \\ &= 0 \end{aligned}$$

if $n - p + n - q > n$, i.e. $p + q < n$. Therefore, in the long exact sequence induced by (2.1), the natural restriction map $H^q(X, \Omega_X^p) \to H^q(Y, \Omega_X^p|_Y)$ is injective for $p + q < n$, and bijective for $p + q < n - 1$.

The map $H^q(X, \Omega_X^p) \to H^q(Y, \Omega_X^p|_Y)$ needs to be composed with the map $H^q(Y, \Omega_X^p|_Y) \to H^q(Y, \Omega_Y^p)$ to get $H^q(X, \Omega_X^p) \to H^q(Y, \Omega_Y^p)$. This part is taken care of by sequence (2.2) and its induced long exact sequence. The restriction $\mathcal{O}_Y(Y)$ of $\mathcal{O}(Y)$ is still positive, as

$$c_1(\mathcal{O}_Y(Y)) = i^* c_1(\mathcal{O}(Y))$$

where i^* is the pullback induced by the inclusion $i : Y \to X$, **which preserves positivity, closedness, and type**. Thus by Kodaira vanishing theorem again,

$$\begin{aligned} H^q(Y, \Omega_Y^{p-1}(-Y)) &= H^{p-1,q}(Y, \mathcal{O}_Y(-Y)) \\ &= H^{n-p+1,n-q}(Y, \mathcal{O}_Y(Y))^* \\ &= H^{n-q}(Y, \Omega^{n-p+1} \otimes \mathcal{O}_Y(Y))^* \\ &= 0 \end{aligned}$$

if $n - p + 1 + n - q > n$, i.e. $p + q - 1 < n$. Therefore, the map $H^q(Y, \Omega_X^p|_Y) \to H^q(Y, \Omega_Y^p)$ is injective for $p + q - 1 < n$, and bijective for $p + q < n$. This completes the proof. □

Remark 2.8. The composition of maps

$$H^q(X, \Omega_X^p) \to H^q(Y, \Omega_X^p|_Y) \to H^q(Y, \Omega_Y^p)$$

consist of pullback and restriction. This coincides with the map $H^q(X, \Omega_X^p) \to H^q(Y, \Omega_Y^p)$ in the bidegree decomposition of $H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})$, which is defined by pullback of forms.

2.3 Variations of Kodaira Vanishing

Slight modification of the proof of Kodaira vanishing theorem yields the following, called Serre's vanishing theorem.

Proposition 2.9 (Serre's Vanishing Theorem)**.** *Let L be a positive line bundle on a compact Kähler manifold X of dimension n. For any holomorphic vector bundle E on X there exists a constant m_0 such that*

$$H^q(X, E \otimes L^m) = 0$$

for all $m \geq m_0$, $q > 0$.

Proof. Since L is positive, we fix Hermitian structure on L such that the curvature of the Chern connection ∇_L on L satisfies

$$\frac{i}{2\pi} F_{\nabla_L} = \omega,$$

where ω is a Kähler form. We endow X with the Kähler structure given by ω. Fix Hermitian structure on E and let ∇_E denote its Chern connection.

The induced connection ∇ on $E \otimes L^m$ is

$$\nabla = \nabla_E \otimes 1 + 1 \otimes \nabla_{L^m},$$

where ∇_{L^m} is induced by ∇_L on L. By Lemma 2.5,

$$\frac{i}{2\pi}\left([\Lambda, F_\nabla]\alpha, \alpha\right) \geq 0,$$

for any $\alpha \in \mathcal{A}^{p,q}(X, E \otimes L^m)$. The curvature on L^m is

$$\frac{i}{2\pi} F_{\nabla_{L^m}} = m \cdot \omega,$$

and

$$\frac{i}{2\pi} F_\nabla = \frac{i}{2\pi} F_{\nabla_E} \otimes 1 + m(1 \otimes \omega).$$

Thus (L_ω denotes the Lefschetz operator on X)

$$\begin{aligned} 0 \leq \frac{i}{2\pi}\left([\Lambda, F_\nabla]\alpha, \alpha\right) &= \frac{i}{2\pi}\left([\Lambda, F_{\nabla_E}]\alpha, \alpha\right) + m\left([\Lambda, L_\omega]\alpha, \alpha\right) \\ &= \frac{i}{2\pi}\left([\Lambda, F_{\nabla_E}]\alpha, \alpha\right) + m(n - (p+q))\|\alpha\|^2 \end{aligned}$$

Apply Cauchy-Schwarz to the first term:

$$|([\Lambda, F_{\nabla_E}]\alpha, \alpha)| \leq \|[\Lambda, F_{\nabla_E}]\| \cdot \|\alpha\|^2.$$

The operator norm $C := \|[\Lambda, F_{\nabla_E}]\| < \infty$ exists by **compactness** of X, and is independent of m. Thus if $C + 2\pi m(n - (p+q)) < 0$, then $\alpha = 0$. In particular, fixing any $m_0 > C/2\pi$, we have

$$H^q(X, \mathcal{K}_X \otimes E \otimes L^m) = 0$$

for all $m \geq m_0$ and all $q > 0$.

Now for any holomorphic vector bundle E, find m_0 with respect to $\tilde{E} := \mathcal{K}_X^* \otimes E$ by the argument above. Then we have

$$H^q(X, E \otimes L^m) = 0$$

for all $m \geq m_0$ and all $q > 0$.

□

We can use Serre's theorem to classify holomorphic vector bundles on $\mathbb{P}^1$.

Corollary 2.10 (Grothendieck Lemma)**.** *Every holomorphic vector bundle E on $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles*

$$\bigoplus_{i=1}^{r} \mathcal{O}(a_i),$$

where the integers $a_1 \geq \cdots \geq a_r$ are uniquely determined by E.

Proof. First suppose E is line bundle. We have shown in ex.3.2.11 that $c_1 : \mathrm{Pic}(\mathbb{P}^1) \to H^2(\mathbb{P}^1, \mathbb{Z})$ is isomorphism. $H^2(\mathbb{P}^1, \mathbb{Z}) \cong H^0(\mathbb{P}^1, \mathbb{Z}) = \mathbb{Z}$ by integration over $\mathbb{P}^1$. We have seen in §4.4 that

$$\int_{\mathbb{P}^1} c_1(\mathcal{O}(1)) = 1.$$

Thus any holomorphic line bundle over $\mathbb{P}^1$ whose first Chern class is $a \in \mathbb{Z}$ must be $\mathcal{O}(a)$.

For arbitrary rank r we prove by induction. First choose a_1 to be the maximal integer a such that $H^0(\mathbb{P}^1, E(-a) = \mathcal{O}(a)^* \otimes E) \neq 0$. Such a_1 must exist. Since $\mathcal{O}(1)$ is positive line bundle, by Serre's vanishing theorem, $H^1(\mathbb{P}^1, E(-a)) = 0$ for $a \ll 0$. By Riemann-Roch formula (1.1) for curves,

$$h^0(\mathbb{P}^1, E(-a)) - h^1(\mathbb{P}^1, E(-a)) = \chi(\mathbb{P}^1, E(-a)) = \deg(E) - a + \mathrm{rank}(E) > 0$$

if $a \ll 0$. Therefore $H^0(\mathbb{P}^1, E(-a) \neq 0$ for $a \ll 0$. To see that the maximal a exists, by Serre's vanishing theorem again,

$$H^0(\mathbb{P}^1, E(-a)) = H^{1,1}(\mathbb{P}^1, E^* \otimes \mathcal{O}(a))^* = H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1} \otimes E^* \otimes \mathcal{O}(a))^* = 0$$

for $a \gg 0$.

Now that we have fixed a_1, pick a non-zero section $0 \neq s \in H^0(\mathbb{P}^1, E(-a_1))$. We can also view s as a sheaf homomorphism $s : \mathcal{O}(a_1) \to E$. We claim that s is in fact a vector bundle homorphism, i.e. a map of constant rank 1. Suppose otherwise s vanishes at some $x \in \mathbb{P}^1$. There is a section $s_x \in H^0(\mathbb{P}^1, \mathcal{O}(1))$ which defines x (e.g. if $x = [0 : z_1]$, then $s_x = z_0 \in \mathbb{C}[z_0, z_1]_1$). Then we can divide s by s_x to get a non-zero section in $H^0(\mathbb{P}^1, E(-a_1 - 1))$, contradiction to the maximality of a_1. Therefore, the holomorphic vector bundle morphism s induces a short exact sequence of holomorphic vector bundles

$$0 \to \mathcal{O}(a_1) \to E \to E_1 = \operatorname{coker}(s) \to 0. \tag{2.3}$$

The sequence splits as complex differentiable vector bundles, and in particular $\operatorname{rank}(E_1) = r - 1$. By induction hypothesis, we have E_1 split as

$$E_1 = \bigoplus_{i>1} \mathcal{O}(a_i).$$

It remains to show that (2.3) splits as holomorphic vector bundles. First we claim that $a_1 \geq a_i$ for all $i > 1$. Suppose otherwise, then $E_1(-a_1 - 1) = \oplus_i \mathcal{O}(a_i - a_1 - 1)$ with $a_i - a_1 - 1 \geq 0$ for some i. Hence $H^0(\mathbb{P}^1, E_1(-a_1 - 1)) \neq 0$. By cor.2.4.2, $H^0(\mathbb{P}^1, \mathcal{O}(-1)) = 0$. Riemann-Roch formula gives $\chi(\mathbb{P}^1, \mathcal{O}(-1)) = 0$, so

$$H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0.$$

Then the long exact sequence associated with the twist of (2.3) by $\mathcal{O}(-a_1 - 1)$

$$0 \to \mathcal{O}(-1) \to E(-a_1 - 1) \to E_1(-a_1 - 1) \to 0$$

yields $H^0(\mathbb{P}^1, E(-a_1 - 1)) \neq 0$, contradiction to the maximality of a_1.

The spitting of (2.3) is equivalent to the splitting of the twist of its dual by $\mathcal{O}(a_1)$:

$$0 \to E_1^*(a_1) \to E^*(a_1) \to \mathcal{O}_{\mathbb{P}^1} \to 0. \tag{2.4}$$

Note that

$$H^1(\mathbb{P}^1, E_1^*(a_1)) = H^1(\mathbb{P}^1, \oplus_{i>1} \mathcal{O}(a_1 - a_i)) = 0,$$

as $a_1 - a_i \geq 0$. To see this, by Riemann-Roch, for each $k \geq 0$,

$$k + 1 = \chi(\mathbb{P}^1, \mathcal{O}(k)) = h^0(\mathbb{P}^1, \mathcal{O}(k)) - h^1(\mathbb{P}^1, \mathcal{O}(k)),$$

and

$$H^0(\mathbb{P}^1, \mathcal{O}(k)) = \mathbb{C}[z_0, z_1]_k \Rightarrow h^0(\mathbb{P}^1, \mathcal{O}(k)) = k + 1.$$

Thus $H^1(\mathbb{P}^1, \mathcal{O}(k)) = 0$ for each $k \geq 0$. Then in the long exact sequence associated with (2.4), the map

$$H^0(\mathbb{P}^1, E^*(a_1)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}))$$

is surjective. The lift of $1 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}))$ in $H^0(\mathbb{P}^1, E^*(a_1))$, considered as a vector bundle homomorphism $\mathcal{O}_{\mathbb{P}^1} \to E^*(a_1)$, splits the sequence (2.4), because its composition with the map $E^*(a_1) \to \mathcal{O}_{\mathbb{P}^1}$ is the identity map on $\mathcal{O}_{\mathbb{P}^1}$.

□

Remark 2.11. In ex.4.4.4 we compute that

$$c_1(\mathbb{P}^1) = c_1(\mathcal{T}_{\mathbb{P}^1}) = 2[\omega].$$

Hence

$$c_1(\Omega_{\mathbb{P}^1}) = -2[\omega] \Rightarrow \int_{\mathbb{P}^1} c_1(\Omega_{\mathbb{P}^1}) = -2.$$

Then by the proof above for the line bundle case,

$$\Omega_{\mathbb{P}^1} \cong \mathcal{O}(-2).$$

Question 2.12. We use the following results from sheaf theory for holomorphic vector bundles in the proof of Grothendieck lemma above.

i) Let $f : E \to F$ be a holomorphic vector bundle homomorphism over a complex manifold X, in particular, f is of constant rank. Then $\mathrm{Im}(f)$ is a holomorphic vector subbundle of F, and we have the short exact sequence of holomorphic vector bundles

$$0 \to \mathrm{Im}(f) \to F \to \mathrm{coker}(f) \to 0.$$

ii) A short exact sequence of holomorphic vector bundles

$$0 \to E \to F \xrightarrow{f} G \to 0$$

may not split. However, the sequence split **if and only if** there exists a holomorphic vector bundle homomorphism $g : G \to F$ such that $f \circ g = \mathrm{Id}_G$.

Grothendieck lemma suggests that there are no interesting vector bundles on $\mathbb{P}^1$ other than line bundles. The situation is more complicated for curves of positive genus and higher diemsnional projective spaces.

Question 2.13. Classification of line bundles, rank two vector bundles on $\mathbb{P}^n$?

We know that $\mathrm{Pic}(\mathbb{P}^n) \cong H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$ by ex.3.2.11, and $c_1(\mathcal{O}(1)) = [\omega_{FS}]$. The map $H^2(\mathbb{P}^n, \mathbb{Z}) \to \mathbb{Z}$ can be realized as

$$[\alpha] \in H^2(\mathbb{P}^n, \mathbb{R}) \mapsto \int_{\mathbb{P}^n} \alpha \wedge \omega_{FS}^{n-1}$$

with

$$\int_{\mathbb{P}^n} \omega_{FS}^n = 1.$$

Then line bundles are also exactly $\mathcal{O}(k), k \in \mathbb{Z}$.

Why is the classification easier when $n \geq 5$?

3 Kodaira Embedding Theorem

This section addresses the following question: when is a compact Kähler manifold projective?

3.1 Preliminaries and Outline of Steps

For preliminary setup of the Kodaira embedding theorem, recall §2.3. Let L be a holomorphic line bundle over a compact complex manifold X. For each choice of basis $\{s_0, ..., s_N\}$ of $H^0(X, L)$, there is a natural rational map

$$\varphi_L : X \setminus \mathrm{Bs}(X) \dashrightarrow \mathbb{P}^N, \quad x \mapsto [s_0(x) : \cdots : s_N(x)], \quad \mathrm{Bs}(X) := \bigcap_{i=1}^N Z(s_i).$$

When is φ_L a closed embedding of X? The choice of the basis only modifies φ_L by an intertible linear isomorphism on $\mathbb{P}^n$, so the answer does not depend on the choice of the basis. We address the question in steps below.

i) First we need $\mathrm{Bs}(X) = \emptyset$. That is, for each $x \in X$, there exists a section $s \in H^0(X, L)$ such that $s(x)$ does not vanish at x. Equivalently, for each $x \in X$, the map

$$H^0(X, L) \to L(x)$$

is surjective. Regarding $L(x)$ as a vector bundle over the 0-dimensional manifold $\{x\}$, we have short exact sequence of sheaves

$$0 \to L \otimes \mathcal{I}_{\{x\}} \to L \to L(x) \to 0$$

by twisting the structure sequence for $\{x\} \subset X$ with L.

ii) Suppose criterion i) is satisfied and φ_L is defined on all of X. We also want φ_L to be injective. This happens if and only if for any $x_1 \neq x_2 \in X$, there exists a section $s \in H^0(X, L)$ with $s(x_1) = 0$ and $s(x_2) \neq 0$. We say that φ_L (or L) separates points.

Combining i) and ii), we see that φ_L is an injective morphism on X if and only if for any $x_1 \neq x_2 \in X$, the map

$$H^0(X, L) \to L(x_1) \oplus L(x_2)$$

is surjective. As above, viewing $\{x_1, x_2\}$ as a 0-manifold, and $L(x_1) \oplus L(x_2)$ a vector bundle over it, we have short exact sequence of sheaves

$$0 \to L \otimes \mathcal{I}_{\{x_1, x_2\}} \to L \to L(x_1) \oplus L(x_2) \to 0.$$

iii) It remains to make sure that φ_L is an immersion. Indeed, recall that **every continuous map from a compact space to a Hausdorff space is both proper and closed; a proper injective immersion is an embedding**. Therefore, for each $x \in X$, we want to check that the differential

$$(d\varphi_L)_x : T_x X \to T_{\varphi_L(x)} \mathbb{P}^N$$

is injective. Choose a section $s_0 \in H^0(X, L)$ with $s_0(x) \neq 0$, and extend it to a basis $\{s_0, ..., s_N\}$ of $H^0(X, L)$ such that $s_i(x) = 0$ for all $i > 0$. Then locally near x we can write φ_L in coordinates as

$$U \subset X \to \mathbb{C}^N, \quad y \mapsto (t_1(y), \dots, t_N(y)), \quad t_i := \frac{s_i}{s_0},$$

where $t_i(x) = 0$ for each $i > 0$. Thus $(d\varphi_L)_x$ is injective if and only if the 1-forms $dt_1, ..., dt_N$ span the cotangent space $\bigwedge_x^1 X$ at x.

We can further translate this condition. Note that the sections $s_1, ..., s_N$ form a basis of the subspace $H^0(X, L \otimes \mathcal{I}_{\{x\}}) \leqslant H^0(X, L)$ of global sections of L that vanish at x. There is a natural map

$$d_x : H^0(X, L \otimes \mathcal{I}_{\{x\}}) \to L(x) \otimes \bigwedge\nolimits_x^1 X$$

defined via any local trivialization $\psi : L|_U \cong U \times \mathbb{C}$: given $s \in H^0(X, L \otimes \mathcal{I}_{\{x\}})$, define

$$\psi_x (d_x(s)) = (d(\psi s))_x,$$

where $\psi s : U \to \mathbb{C}$. To see that d_x is defined independent of the choice of ψ, suppose $\psi' = \lambda \psi$ is another local trivialization, then we get the same definition for d_x since $s(x) = 0$.

Now consider dt_i. Since $t_i(x) = 0$, again by product rule we have

$$(dt_i)_x = (\psi s_0)^{-1} (d(\psi s_i))_x.$$

Therefore, $(d\varphi_L)_x$ is injective if and only if $d_x : H^0(X, L \otimes \mathcal{I}_{\{x\}}) \to L(x) \otimes \bigwedge_x^1 X$ is surjective. As before, d_x is induced by a short exact sequence

$$0 \to L \otimes \mathcal{I}_{\{x\}}^2 \to L \otimes \mathcal{I}_{\{x\}} \to L(x) \otimes \bigwedge\nolimits_x^1 X \to 0.$$

This is the twist by L of the short exact sequence

$$0 \to \mathcal{I}_{\{x\}}^2 \to \mathcal{I}_{\{x\}} \xrightarrow{d} \bigwedge\nolimits_x^1 X \to 0.$$

Indeed, a holomorphic function f near x satisfies $f(x) = df(x) = 0$ if and only if it vanishes at x of order at least 2. Thus there is canonical isomorphism $\mathcal{I}_{\{x\}}/\mathcal{I}_{\{x\}}^2 \overset{d}{\cong} \bigwedge_x^1 X$.

In conclusion, the complete linear system $H^0(X, L)$ induces a closed embedding $\varphi_L : X \hookrightarrow \mathbb{P}^N$ if and only if the global sections of L separates points and tangent directions in T_xX.

Question 3.1. Why is $H^0(X, L \otimes \mathcal{I}_{\{x\}})$ the space of global sections of L that vanish at x? A section s can be thought of as a map from each $x \in X$ to its stalk. See def.B.0.25. The stalks are $\left(L \otimes \mathcal{I}_{\{x\}}\right)_y = L_y \otimes_{\mathcal{O}_{X,y}} \left(\mathcal{I}_{\{x\}}\right)_y$ for each y. Then near x we can write $s \in H^0(X, L)$ as a product of a section of L with a holomorphic function vanishing at x. This makes s a section of $L \otimes \mathcal{I}_{\{x\}}$.

Sheaves only concern what happens locally!

3.2 Statement and Proof

We are now ready to state and prove the Kodaira embedding theorem.

Definition 3.2. Let L be a holomorphic line bundle on a compact complex manifold X. L is called an **ample bundle** if there exists some $k > 0$ such that $H^0(X, L^k)$ defines a closed embedding $\varphi_{L^k} : X \hookrightarrow \mathbb{P}^N$.

Theorem 3.3. *Let X be a compact Kähler manifold. A line bundle L on X is positive if and only if L is ample. If such a line bundle exists, then X is projective.*

To prove the Kodaira embedding theorem, we first study the positivity of line bundles under blow-ups.

Lemma 3.4. *Let X be a complex manifold of dimension n and L a postive line bundle on X. Let $\sigma : \hat{X} \to X$ be the blow-up of X along a finite number of points $x_1, ..., x_l \in X$, and let $E_j := \sigma^{-1}(x_j) \cong \mathbb{P}(T_xX)$ be the exceptional divisors for each $j = 1, ..., l$. Then for anu holomorphic line bundle M on X and integers $n_1, ..., n_k > 0$, the line bundle*

$$\sigma^*(L^k \otimes M) \otimes \mathcal{O}(-\sum_j n_j E_j)$$

on $\hat{X}$ is positive for $k \gg 0$.

Proof. For each j, we can pick a neighborhood $x_j \in U_j \subset X$ such that the blow up can be seen as the incidence variety $\hat{U}_j = \mathcal{O}(-1) \subset U_j \times \mathbb{P}^{n-1}$. Recall from prop.2.5.6 that $\mathcal{O}(E_j)$ is isomorphic to $p_j^*\mathcal{O}(-1)$, where $p_j : \hat{U}_j \to \mathbb{P}^{n-1}$ is the second projection. More specifically, we can identify each fiber as

$$\mathcal{O}(E)|_{(z,l)} = \{\lambda(l_1, ..., l_n) \mid \lambda \in \mathbb{C}\}.$$

Since we have the natural Fubini-Study hermitian structure on $\mathcal{O}(1)$, we endow $\mathcal{O}(-E_j)$ with the pullback hermitian structure. Then by a partition of unity we can glue the n_j-th powers of the hermitian structures on $\mathcal{O}(-E_j)$ to get a hermitian structure on $\mathcal{O}(-n_jE_j)$. Hence locally near each E_j, the curvature is

$$\frac{i}{2\pi} F_\nabla = n_j \cdot p_j^* \omega_{FS},$$

where ω_{FS} is the Fubini-Study Kähler form on $\mathbb{P}^{n-1}$. Therefore, F_∇ is locally semi-positive around each E_j, and strictly positive for all tangent directions of E_j, due to definition of p_j.

Choose real (1, 1)-forms α and β such that $[\alpha] = c_1(L)$ and $[\beta] = c_1(M)$, where α is positive by positivity of L. Then the real (1, 1)-form

$$\sigma^*(k \cdot \alpha + \beta) + \frac{i}{2\pi} F_\nabla$$

representing $c_1\left(\sigma^*(L^k \otimes M) \otimes \mathcal{O}(-\sum_j n_j E_j)\right)$ is positive on $\hat{X}$ for $k \gg 0$ by compactness argument, as σ is the first projection, which **takes care of all tangent directions to $\hat{X}$ away from E_j and does not contribute to directions tangent to E_j.**

□

Proof of the Kodaira Embedding Theorem. ⇒: Suppose L is ample, and for some $k > 0$ we have closed embedding $\varphi_{L^k} : X \to \mathbb{P}^N$. Recall from §2.3 that $\varphi_{L^k}^* \mathcal{O}(1) \cong L^k$, so

$$c_1(L) = \frac{1}{k}[\varphi_{L^k}^* \omega_{FS}],$$

where $\varphi_{L^k}^* \omega_{FS}$ is positive because φ_{L^k} is embedding. Therefore, L is positive line bundle.

⇐: Suppose L is positive. We first show the injectivity of φ_{L^k} for some high power k. Let $\sigma : \hat{X} \to X$ be the blow-up of X along $x \in X$, and let $E := \sigma^{-1}(x)$ be the exceptional divisor. We have commutative diagram

$$\begin{array}{ccc} H^0(X, L^k) & \longrightarrow & L^k(x) \\ \sigma^* \downarrow & & \downarrow \cong \\ H^0(\hat{X}, \sigma^* L^k) & \longrightarrow & H^0(E, \mathcal{O}_E) \otimes L^k(x) \end{array}$$

The top horizontal map is evaluation at x. The left vertical map by pullback is injective since σ is surjective.

We claim that the left vertical map is in fact bijective. If $\dim_{\mathbb{C}} X = 1$, then the blow up is trivial. If $\dim_{\mathbb{C}} X \geq 2$, we decompose the map as

$$H^0(X, L^k) \to H^0(X \setminus \{x\}, L^k) \cong H^0(\hat{X} \setminus E, \sigma^* L^k) \leftarrow H^0(\hat{X}, \sigma^* L^k).$$

The first map is bijective by ex.2.2.6. The third map is injective by Riemann extension theorem. Hence the map $H^0(X, L^k) \to H^0(\hat{X}, \sigma^* L^k)$ is surjective.

Twisting the structure sequence for $E \subset \hat{X}$ by $\sigma^*(L^k)$, we have short exact sequence

$$0 \to \sigma^*(L^k) \otimes \mathcal{O}(-E) \to \sigma^*(L^k) \to \sigma^*(L^k)|_E \cong L^k(x) \otimes \mathcal{O}_E \to 0.$$

Thus the cokernel of the bottom map $H^0(\hat{X}, \sigma^* L^k) \to H^0(E, \mathcal{O}_E) \otimes L^k(x)$ can be viewed as contained in $H^1(\hat{X}, \sigma^* L^k \otimes \mathcal{O}(-E))$.

Let $\dim_{\mathbb{C}} X = n$. Prop.2.5.5 shows

$$\mathcal{K}_{\hat{X}} \cong \sigma^* \mathcal{K}_X \otimes \mathcal{O}((n-1)E).$$

Thus by lemma above, the line bundle

$$L' := \sigma^* L^k \otimes \mathcal{K}_{\hat{X}}^* \otimes \mathcal{O}(-E) = \sigma^*(L^k \otimes \mathcal{K}_X^*) \otimes \mathcal{O}(-nE)$$

is positive for $k \gg 0$. Hence by Kodaira vanishing theorem, for $k \gg 0$, we have

$$H^1(\hat{X}, \sigma^* L^k \otimes \mathcal{O}(-E)) = H^1(\hat{X}, \mathcal{K}_{\hat{X}} \otimes L') = H^1(\hat{X}, \Omega_{\hat{X}}^n \otimes L') = 0.$$

In conclusion, the map

$$H^0(\hat{X}, \sigma^* L^k) \to H^0(E, \mathcal{O}_E) \otimes L^k(x)$$

and hence the map

$$H^0(X, L^k) \to L^k(x)$$

in the commutative diagram is surjective for $k \gg 0$. By discussion i) in the beginning of this section, $x \notin \mathrm{Bs}(L^k)$ for $k \gg 0$.

We have not yet concluded that $\mathrm{Bs}(L^k)=\emptyset$ for some $k>0$, for in the above the number k may depend on x. To see this, notice that there is a decreasing sequence (by compactness of X)

$$\mathrm{Bs}(L)\supset\cdots\supset\mathrm{Bs}(L^{2^l})\supset\mathrm{Bs}(L^{2^{l+1}})\supset\dots$$

by the map

$$H^0(X,L^{2^l})\to H^0(X,L^{2^{l+1}}),\quad s\mapsto s\otimes s.$$

From last paragraph, we have

$$\bigcap_{l=0}^{\infty}\mathrm{Bs}(L^{2^l})=\emptyset.$$

Thus by compactness of X, there exists some $l\gg 0$ such that $\mathrm{Bs}(L^{k:=2^l})=\emptyset$. This checks step i): the map φ_{L^k} is defined everywhere on X for some $k\gg 0$.

To check condition ii), we analogously blow up X along two distinct points $x_1\neq x_2\in X$ and consider $\sigma^*L^k\otimes\mathcal{O}(-E_1-E_2)$, to show that the map

$$H^0(X,L^k)\to L^k(x_1)\oplus L^k(x_2)$$

is surjective for some $k\gg 0$. Again by compactness of X (and hence $X\times X$), there is universal $k\gg 0$ such that $H^0(X,L^k)\to L^k(x_1)\oplus L^k(x_2)$ is surjective for all $x_1\neq x_2$.

It remains to check condition iii), that the map

$$d_x: H^0(X,L\otimes\mathcal{I}_{\{x\}})\to L(x)\otimes{\bigwedge}_x^1 X$$

is surjective. Recall the short exact sequence

$$0\to\mathcal{I}_{\{x\}}^2\to\mathcal{I}_{\{x\}}\xrightarrow{d}{\bigwedge}_x^1 X\to 0. \tag{3.1}$$

Twist the structure sequence for $E\subset\hat{X}$ by $\mathcal{O}(-E)$, we get short exact sequence

$$0\to\mathcal{O}(-2E)\to\mathcal{O}(-E)\to\mathcal{O}_E(-E)\to 0. \tag{3.2}$$

Recall by lem.2.3.22 the map $\mathcal{O}(-E)\to\mathcal{O}_{\hat{X}}$ which is injective and whose image is isomorphic to the ideal sheaf $\mathcal{I}_E$ of holomorphic functions on $\hat{X}$ vanishing on E. Each holomorphic function on X vanishing at x pulls back by σ to vanish on E. Thus we have the vertical maps in the commutative diagram

$$\begin{array}{ccc}\sigma^*\mathcal{I}_{\{x\}}^2 & \longrightarrow & \sigma^*\mathcal{I}_{\{x\}}\\ \downarrow & & \downarrow\cong\\ \mathcal{O}(-2E) & \longrightarrow & \mathcal{O}(-E)\end{array}$$

connecting the two short exact sequences give above. Twisting sequence (3.1) by L^k and sequence (3.2) by σ^*L^k, and passing to the quotients, we get a commutative diagram

$$\begin{array}{ccc}H^0(X,L^k\otimes\mathcal{I}_{\{x\}}) & \longrightarrow & L^k(x)\otimes{\bigwedge}_x^1 X\\ \cong\downarrow\sigma^* & & \downarrow\cong\\ H^0(\hat{X},\sigma^*L^k(-E)) & \longrightarrow & L^k(x)\otimes H^0(E,\mathcal{O}_E(-E))\end{array}$$

The top map evaluates the L^k part at x and takes differential operator d on the $\mathcal{I}_{\{x\}}$ part. This map is well-defined on the tensor $L^k\otimes\mathcal{I}_{\{x\}}$ because functions in $\mathcal{I}_{\{x\}}$ vanish at x. For the left vertical map, recall that the map $\mathcal{O}(-E) \to \mathcal{O}_{\hat{X}}$ is the dual of the map $\mathcal{O}_{\hat{X}} \to \mathcal{O}(E)$ defined by a non-zero section $s \in H^0(\hat{X}, \mathcal{O}(E))$ with $Z(s) = E$. **Thus the left vertical map essentially divides the pullback function by the section s defining E.** The bottom map is the restriction of a section in $H^0(\hat{X}, \sigma^* L^k(-E))$ to E. For a neighborhood $x \in U \subset X$ with local trivialization of L^k, we have the associated local trivialization of $\sigma^* L^k$ on $\hat{U} := \sigma^{-1}(U) \subset \hat{X}$ containing E. Then each section of $\sigma^* L^k$ restricted to the compact set E is a holomorphic function in the local trivialization, which must hence be a constant. This recovers the image in $L^k(x)$ by the local triviliazation of L^k, and the $\mathcal{O}(-E)$ part is simply restriction to E. For the right vertical map, we claim that

$$H^0(E, \mathcal{O}_E(-E)) \cong \bigwedge\nolimits_x^1 X. \tag{3.3}$$

For each holomorphic function f defined near x and vanishing at x, $\sigma^* f$ vanishes on E. Thus it defines a section $\sigma^* f \otimes s^*$ of $\mathcal{O}(-E)$ near E as in the vertical map. Then under the identification (3.3), $(\sigma^* f \otimes s^*)\,|_E \in H^0(E, \mathcal{O}_E(-E))$ corresponds to the differential $df_x \in \bigwedge_x^1 X$. This is discussed in GH pg.185, and can be proved using the local coordinates defined in that book, combined with the previous result that

$$H^0(\mathbb{P}^k, \mathcal{O}(1)) \cong \mathbb{C}[z_0, ..., z_k]_1.$$

Here $E \cong \mathbb{P}(T_x X) \cong \mathbb{P}^{n-1}$, and $\mathcal{O}_E(E) \cong \mathcal{O}(-1)$. Indeed this identification is exactly the combination of the top, left, and bottom maps, and hence the diagram commutes.

The left vertical map is bijective, with analogous proof to the one for $H^0(X, L^l) \cong H^0(\hat{X}, \sigma^* L^k)$ above. Thus as above, the surjectivity of the top horizontal map

$$H^0(X, L^k \otimes \mathcal{I}_{\{x\}}) \xrightarrow{d_x} L^k(x) \otimes \bigwedge\nolimits_x^1 X$$

is equivalent to the sujectivity of the bottom horizontal map

$$H^0(\hat{X}, \sigma^* L^k(-E)) \to L^k(x) \otimes H^0(E, \mathcal{O}_E(-E)) = H^0(E, \sigma^* L^k \otimes \mathcal{O}_E(-E)),$$

which is further equivalent to

$$H^1(\hat{X}, \sigma^* L^k \otimes \mathcal{O}(-2E)) = 0$$

by the long exact sequence associated to the twist of (3.2) by $\sigma^* L^k$. Using

$$\mathcal{K}_{\hat{X}} \cong \sigma^* \mathcal{K}_X \otimes \mathcal{O}((n-1)E),$$

the lemma above, and Kodaira vanishing theorem again, we see that indeed

$$\begin{aligned} H^1(\hat{X}, \sigma^* L^k \otimes \mathcal{O}(-2E)) &= H^1(\hat{X}, \mathcal{K}_{\hat{X}} \otimes \left(\mathcal{K}_{\hat{X}}^* \otimes \sigma^* L^k \otimes \mathcal{O}(-2E)\right)) \\ &= H^1(\hat{X}, \mathcal{K}_{\hat{X}} \otimes \left(\sigma^* \left(L^k \otimes \mathcal{K}_X^*\right) \otimes \mathcal{O}((-n-1)E)\right)) \\ &= 0 \end{aligned}$$

for $k \gg 0$. Therefore, for each $x \in X$, there exists some $k \gg 0$ such that $d\varphi_{L^k}$ satisfies condition i) and ii) and is injective at x, or equivalently, the map

$$H^0(X, L^k \otimes \mathcal{I}_{\{x\}}) \xrightarrow{d_x} L^k(x) \otimes \bigwedge\nolimits_x^1 X$$

is surjective.

We are left to prove the existence of universal $k \gg 0$ for all x as above, with slight modification. Define

$$S(L^k) := \{x \in X \mid (d\varphi_{L^k})_x \text{ is NOT injective}\},$$

and consider the sequence

$$S(L^{2_0^l}) \supset S(L^{2^{l_0+1}}) \supset \dots$$

where we start from some $l_0 \gg 0$ such that conditions i) and ii) are satisfied for all higher powers $L^{2^{l\geq l_0}}$. We claim that the sequence is indeed decreasing. Suppose $x \in S(L^{2^{l+1}}) \setminus S(L^{2^l})$ for some $l \geq l_0$. Then the map

$$H^0(X, L^{2^l} \otimes \mathcal{I}_{\{x\}}) \to L^{2^l}(x) \otimes \bigwedge_x^1 X$$

is surjective. We have commmutative diagram

$$\begin{array}{ccc} H^0(X, L^{2^l}) \otimes H^0(X, L^{2^l} \otimes \mathcal{I}_{\{x\}}) & \xrightarrow{\mathrm{eval}_x \otimes d_x} & L^{2^l}(x) \otimes L^{2^l}(x) \otimes \bigwedge_x^1 X \\ \downarrow & & \downarrow = \\ H^0(X, L^{2^{l+1}} \otimes \mathcal{I}_{\{x\}}) & \xrightarrow[d_x]{} & L^{2^{k+1}}(x) \otimes \bigwedge_x^1 X \end{array}$$

where the left vertical map takes $\mathbb{C}$-tensor product of sections to the section tensor product of those two sections. The map

$$H^0(X, L^{2^l}) \xrightarrow{\mathrm{eval}_x} L^{2^l}(x)$$

is surjective (this is condition i) we checked above). Thus the top horizontal map in the diagram above is surjective. By commutativity, the bottom map is also surjective. This yields a contradiction to $x \in S(L^{2^{l+1}})$, and we conclude $S(L^{2^{l+1}}) \subset S(L^{2^l})$ for each $l \geq l_0$. Beyond this point we proceed exactly as above for $\mathrm{Bs}(L)$, and conclude by compactness of X that $S(L^{2^l}) = \emptyset$ for some $l \geq l_0$.

For this l, we finally get a closed embedding

$$\varphi_{L^{2^l}} : X \hookrightarrow \mathbb{P}^N.$$

Therefore, the positive line bundle L is ample.

□

3.3 Applications

As a first application, we can tell the projectivity of a compact Kähler manifold by the position of the Kähler cone $K_X \subset H^2(X, \mathbb{R})$ relative to the integral lattice $H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R})$. Recall that we define the **Kähler cone** to be the set of all classes of Kähler forms in $H^2(X, \mathbb{R})$. By Hodge theory, this is contained in $H^{1,1}(X) \cap H^2(X, \mathbb{R})$ if we embed all cohomology groups in $H^2(X, \mathbb{C})$.

Corollary 3.5. *A compact Kähler manifold X is projective if and only if $K_X \cap H^2(X, \mathbb{Z}) \neq \emptyset$.*

Proof. $\Rightarrow$: Suppose X is compact Kähler and projective. Then there is closed embedding $\varphi : X \to \mathbb{P}^N$. We claim that the class of the pullback of the Fubini-Study metric on $\mathbb{P}^N$ is contained in $K_X \cap H^2(X, \mathbb{Z})$:

$$[\varphi^* \omega_{FS}] \in K_X \cap H^2(X, \mathbb{Z}).$$

To see this, let $\mathcal{O}(1)$ denote the dual bundle of incidence variety on $\mathbb{P}^N$. The pullback bundle $\varphi^*\mathcal{O}(1)$ is a positive line bundle because

$$c_1(\varphi^*\mathcal{O}(1)) = [\varphi^*\omega_{FS}],$$

where $\varphi^*\omega_{FS}$ is a closed, positive, real (1, 1)-form because φ is an embedding. Hence $\varphi^*\omega_{FS}$ is a Kähler form, and $[\varphi^*\omega_{FS}] \in K_X$. We know that first Chern classes are always contained in $H^2(X, \mathbb{Z})$ by prop.4.4.12. This proves the claim.

$\Leftarrow$: Pick $[\alpha] \in K_X \cap H^2(X, \mathbb{Z})$. By Lefschetz theorem, the first Chern class map

$$c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{C})$$

has image contained in

$$H^{1,1}(X, \mathbb{Z}) := \mathrm{Im}\{H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})\} \cap H^{1,1}(X)$$

and is surjective onto $H^{1,1}(X,\mathbb{Z})$. Thus picking a class $c \in K_X \cap H^2(X,\mathbb{Z})$, we have $c \in H^{1,1}(X,\mathbb{Z})$, so that $c = c_1(L)$ for some line bundle L on X. The class $c = c_1(L)$ can be represented by a Kähler form $\omega \in c$ that is closed, positive and real of type (1, 1). Thus L is positive line bundle. By Kodaira embedding theorem, X is projective. □

Motivated by the corollary above, we define **Hodge class** to be any class in $K_X \cap H^2(X,\mathbb{Z})$. Hence a compact Kähler manifold is projective if and only if it admits a Hodge class.

For a special case, consider complex tori.

Corollary 3.6. *Let $X := V/\Gamma$ be a complex torus. Then X is projective if and only if X admits a* ***Riemann form****, i.e. an alternating $\mathbb{R}$-bilinear form $\omega : V \times V \to \mathbb{R}$ such that*

i) $\omega(iu, iv) = \omega(u, v)$,

ii) $\omega(\cdot, i(\cdot))$ *is positive definite, and*

iii) $\omega(u, v) \in \mathbb{Z}$ *if* $u, v \in \Gamma$.

Proof. $\Leftarrow$: Suppose we have a Riemann form ω. Extend this form trivially to be a constant real 2-form on V under the trivial identification of $TV = V \times V$. Since the 2-form is invariant under Γ, we get a closed real 2-form α on X. $V \cong \mathbb{C}^n$ has natural complex structure I, which coincides with multiplciation by i. Thus condition i) implies that α is of type (1, 1). Condition ii) implies that α is positive real (1, 1)-form. Hence α is a Kähler form. Condition iii) implies that $[\alpha] \in H^2(X,\mathbb{Z})$, as we can identify the image of $H^2(X,\mathbb{Z}) \subset H^2(X,\mathbb{R})$ using cellular cohomology: $H^2(X,\mathbb{Z}) \cong \mathbb{Z}^{\binom{2n}{2}}$ is the space of $\mathbb{Z}$-valued functions on 2-cells, so $[\alpha] \in H^2(X,\mathbb{Z})$ as the element sending the 2-cells generated by $e_i, e_j \in \Gamma$ to $\alpha(e_i, e_j)$. Therefore, $[\alpha] \in H^2(X,\mathbb{C})$ is a Hodge class, and by Corollary 3.5, X is projective.

$\Rightarrow$: Suppose X is projective, then we can pick a Kähler form α representing a Hodge class as in the proof of Corollary 3.5. Then α is an alternating $\mathbb{R}$-bilinear form satisfying i) and ii) at every point $x \in X$ under the global trivialization $TX \cong X \times V$. Endow X with the trivial hermitian metric g and associated Kähler structure. For this metric the harmonic forms are constant linear combinations of dx_I by maximum principle. Thus we can represent the hodge class in $H^2(X,\mathbb{R})$ by a constant real harmonic 2-form β, and we let $\omega = \beta(x)$ for any $x \in X$. Since α is of type (1, 1), by Hodge decomposition (Cor.3.2.12), β is of type (1, 1). Thus ω checks condition i).

To check condition ii), write $\alpha = \beta + d\gamma$ for some real 1-form γ. Then notice that

$$\int_X \alpha(u,v)\,\mathrm{Vol} = \int_X \beta(u,v)\,\mathrm{Vol} \Leftarrow \int_X d\gamma(u,v)\,\mathrm{Vol} = 0$$

for any constant vector fields $u, v \in V$. This can be easily checked using Stokes' theorem and writing $\gamma = \gamma^i dx_i$. Thus the positive definiteness of α ensures the positive definiteness of β everywhere.

Embedding $H^2(X,\mathbb{Z}) \subset H^2(X,\mathbb{R})$ as above, since $[\beta] \in H^2(X,\mathbb{Z})$, we see that β assigns integer value to each 2-cell given by $e_i, e_j \in \Gamma$. This checks condition iii).

□

Example 3.7. Any compact complex curve is projective. Since $\dim_{\mathbb{C}} X = 1$, we see that any Hermitian metric on X is Kähler. Fix any $x \in X$, and consider the line bundle $\mathcal{O}(\{x\})$. We have

$$c_1(\mathcal{O}(\{x\})) = [x] \in H^2(X,\mathbb{R}) \cong \mathbb{R}.$$

Thus $[x] = c[\omega]$ for some constant $c \in \mathbb{R}$ and a Kähler form ω. Then for the function $f \equiv 1$ on X, $f \in H^0(X,\mathbb{R})$,

$$c|X| = c\int_X f\omega = \int_X f[x] = \int_x f = f(x) = 1.$$

Indeed $|X| = \int_X \omega > 0$, so $c > 0$. Therefore, $\mathcal{O}(\{x\})$ is a positive line bundle on X. By Kodaira embedding theorem, X is projective.

Another proof uses Corollary 3.5. Since $\dim_{\mathbb{C}} X = 1$, we have $H^{1,1}(X) = H^2(X, \mathbb{C}) = \mathbb{C}$. Recall from ex.3.1.12 that K_X is an open convex cone in $H^{1,1}(X) \cap H^2(X, \mathbb{R})$, which in this case is a real line in $\mathbb{C}$. $H^2(X, \mathbb{Z})$ is a lattice on this real line. Thus $K_X \cap H^2(X, \mathbb{Z})$ is non-empty.

Example 3.8. As a generalization of the second proof above, every compact Kähler manifold X with $H^{0,2}(X) = 0$ is projective. Taking complex conjugate we see $H^{2,0}(X) = 0$, and hence $H^2(X, \mathbb{C}) = H^{1,1}(X)$. Thus K_X is open convex cone in $H^2(X, \mathbb{R})$, which must intersect the lattice of image $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$.

Combining Hirzebruch-Riemann-Roch, Kodaira vanishing, and Kodaira embedding, we answer a question raised in §2.3:

Corollary 3.9. *Let X be a projective manifold. Then the natural homomorphism*

$$\mathcal{O} : \mathrm{Div}(X) \to \mathrm{Pic}(X)$$

is surjective.

Proof. First observe that X is compact Kähler. For each ample line bundle L and any line bundle M, by Serre vanishing theorem (Proposition 2.9), there exists a constant m_0 such that

$$H^q(X, M \otimes L^m) = 0$$

for all $m \geq m_0$, $q > 0$. Thus the Euler-Poincaré characteristic reads

$$\chi(X, M \otimes L^k) = h^0(X, M \otimes L^k)$$

for $k \gg 0$. By Hirzebruch-Riemann-Roch, we compute

$$\begin{aligned} \chi(X, M \otimes L^k) &= \int_X [\mathrm{ch}(M)\,\mathrm{td}(X)] e^{kc_1(L)} \\ &= \frac{1}{n!} \int_X c_1(L)^n k^n + \alpha_{n-1} k^{n-1} + \cdots + \alpha_1 k + \alpha_0, \end{aligned}$$

which is a polynomial in k of degree $n = \dim_{\mathbb{C}} X$. The leading coefficient is positive, because L is positive line bundle by Kodaira embedding theorem. More specifically, we can represent $c_1(L)$ by a positive real (1, 1)-form, and compute in local coordinates using the linear algebra lemma that the determinant of a positive definite matrix is positve.

Therefore, $H^0(X, M \otimes L^k) \neq 0$ for $k \gg 0$. In particular, for $M = \mathcal{O}_X$, we have $H^0(X, L^k) \neq 0$ for $k \gg 0$. Using the map

$$Z : H^0(X, L) \setminus \{0\} \to \mathrm{Div}(X),$$

and prop.2.3.18, we see that $M \otimes L^k$ and L^k are both contained in the image of $\mathcal{O} : \mathrm{Div}(X) \to \mathrm{Pic}(X)$. Since $\mathcal{O}$ is homomorphism, M is also contained in the image of $\mathcal{O}$. □

Remark 3.10.

1. The fact $H^0(X, L^k) \neq 0$ for some $k > 0$ used in the proof above also follows immediately from L being ample.

2. In the proof above we only use the easy direction of Kodaira embedding theorem: an ample line bundle is positive on compact Kähler manifold.

3. Suppose $0 \neq s_1 \in H^0(X, M \otimes L^k)$, $0 \neq s_2 \in H^0(X, L^k)$, such that $\mathcal{O}(Z(s_1)) \cong M \otimes L^k$, $\mathcal{O}(Z(s_2)) \cong L^k$. Then s_1/s_2 is a non-trivial global meromorphic section of M. Thus for a projective manifold, any holomorphic line bundle admits a non-trivial meromorphic section.

We can apply the corollary above to the Neron-Severi group

$$NS(X) = \mathrm{Im}\Big\{\mathrm{Pic}(X) \xrightarrow{c_1} H^{1,1}(X,\mathbb{Z})\Big\} = H^{1,1}(X,\mathbb{Z})$$

defined in §3.3. Suppose X is projective. Since $c_1(\mathcal{O}(D)) = [D]$ for any divisor $D \in \mathrm{Div}(X)$, $NS(X)$ is spanned by the fundamental classes of divisors. We can further deduce that

$$NS(X) \cong \mathrm{Pic}(X)/NT(X)$$

where $NT(X) \leqslant \mathrm{Pic}(X)$ is the subgroup of numberically trivial line bundles: line bundles which are of degree zero on any curve $C \subset X$. Indeed, for each $L \in \ker c_1$, by ex.4.4.1,

$$\deg(L|_C) = \deg(i^*L) = \int_C c_1(i^*L) = \int_C i^*c_1(L) = 0.$$

Hence $\ker c_1 \subset NT(X)$.

We now show $NT(X) \subset \ker c_1$. Let L be a line bundle that is numerically trivial, i.e.

$$\int_X c_1(L) \wedge [C] = 0$$

for any curve $C \subset X$. Let $D \in \mathrm{Div}(X)$ be any hypersurface. Let ω on X be the pullback of the Fubini-Study metric on $\mathbb{P}^N$. Then there exists some non-zero integer $0 \neq \lambda \in \mathbb{Z}$ such that $\lambda[\omega]^{n-2} \in H^{2n-4}(X,\mathbb{Z})$ is the fundamental class of a closed analytic subvariety of dimension 2 transversal to D. To see this, recall that $[\omega_{FS}]^{n-2}$ is a generator of $H^{2n-4}(\mathbb{P}^N,\mathbb{Z}) \cong \mathbb{Z}$, so that $\lambda[\omega_{FS}]^{n-2}$ is the fundamental class of a subvariety of codimension $n-2$ (transversal to X) for some $0 \neq \lambda \in \mathbb{Z}$. Then $[D] \wedge \lambda[\omega]^{n-2} \in H^{n-1,n-1}(X,\mathbb{Z})$ is the fundamental class of their intersection, hence a curve on X. Thus we have

$$\int_X c_1(L) \wedge [D] \wedge \omega^{n-2} = 0, \quad \text{for any divisor } D \in \mathrm{Div}(X).$$

Since $c_1(L) \in H^{1,1}(X,\mathbb{Z})$, by Hard Lefschetz, we can write

$$c_1(L) = [\alpha] + c[\omega],$$

for some $c \in \mathbb{R}$ and $[\alpha] \in H^{1,1}(X,\mathbb{R})$ primitive. Then $c_1(L) + n[\omega] = [\alpha] + (c+n)[\omega] \in H^{1,1}(X,\mathbb{Z})$, for an integer $n \in \mathbb{Z}$ to be determined. By Corollary 3.9, there exists some divisor D such that $[D] = c_1(L) + n[\omega]$. Thus

$$\begin{aligned} 0 &= \int_X ([\alpha] + c[\omega]) \wedge ([\alpha] + (c+n)[\omega]) \wedge [\omega]^{n-2} \\ &= \int_X [\alpha] \wedge [\alpha] \wedge [\omega]^{n-2} + c(c+n)\int_X \omega^n && [\alpha] \text{ primitive.} \end{aligned}$$

If we pick $n \in \mathbb{Z}$ such that $c(c+n) \leq 0$, then we must have $[\alpha] = 0$, for otherwise $\int_X [\alpha] \wedge [\alpha] \wedge [\omega]^{n-2} < 0$ by Hodge-Riemann bilinear relation. Now $c(c+n)\int_X \omega^n = 0$ for all $n \in \mathbb{Z}$ such that $c(c+n) \leq 0$, so we must have $c = 0$. Therefore, $c_1(L) = 0$, and $NT(X) = \ker c_1$.