Notes on *Extremal Kähler Metrics* by Gábor Székelyhidi

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1 Chapter 2: Elliptic Operators on Compact Manifolds

We collect some important analytic lemmas for future use.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set, and $u_k : \Omega \to \mathbb{R}$ a sequence of functions uniformly bounded in $C^{k,\alpha}$. Then there is a subsequence of u_k which is convergent in $C^{l,\beta}$ for any l, β such that $l + \beta < k + \alpha$.*

This is an easy consequence of Arzela-Ascoli.

From now on let L denote a uniformly elliptic second-order differential opeartor with smooth coefficients on a bounded domain $\Omega \subset \mathbb{R}^n$

$$L(f) = \sum_{j,k} a_{jk} \frac{\partial^2 f}{\partial x^j \partial x^k} + \sum_l b_l \frac{\partial f}{\partial x^l} + cf.$$

Uniform ellipticity means: there exists some $\lambda, \Lambda > 0$ such that

$$\lambda |v|^2 \leq \sum_{j,k} a_{jk}(x) v^j v^k \leq \Lambda |v|^2,$$

for all $x \in \Omega$. Then we have Schauder estimates:

Theorem 1.2 (Local Schauder Estimates)**.** *Suppose $\Omega' \subset \Omega$ is a smaller domain with $d(\Omega', \partial\Omega) > 0$. For each $k \in \mathbb{N}$ and $\alpha \in (0,1)$, there is a constant $C = C(k, \alpha, \Omega', \Omega, L)$ such that if $f, g : \Omega \to \mathbb{R}$ satisfies*

$$L(f) = g,$$

then

$$\|f\|_{C^{k+2,\alpha}(\Omega')} \leq C \left(\|g\|_{C^{k,\alpha}(\Omega)} + \|f\|_{C^0(\Omega)} \right).$$

Remark 1.3. More precisely, C depends on the $C^{k,\alpha}$-norms of the coefficients of L and the ellipticity constants λ, Λ. Regularity theory ensures that if $f \in C^2(\Omega)$ and the coefficients of L and g are in $C^{k,\alpha}(\Omega)$, then $f \in C^{k+2,\alpha}(\Omega)$, so that the local estimate above makes sense.

We can also control the C^0-norm of f by the L^1-norm of f and the C^α-norm of $g = L(f)$. Hence,

Theorem 1.4 (Local Schauder Estimates)**.** *Under the same conditions as above,*

$$\|f\|_{C^{k+2,\alpha}(\Omega')} \leq C \left(\|g\|_{C^{k,\alpha}(\Omega)} + \|f\|_{L^1(\Omega)} \right).$$

We now move on to compact manifolds. Let (M, g) be a compact Riemannian manifold. There are two ways to define Hölder spaces $C^{k,\alpha}(M, g)$ for tensor fields on M.

First, we can fix a priori a finite cover of M by coordinate charts, and treat each tensor T locally as a collection of component functions in each coordinate chart. The $C^{k,\alpha}$-norm of T is then the supremum of the $C^{k,\alpha}$-norms of the components of T over each coordinate chart.

To avoid using coordinate charts, we can define C^{α}-seminorm of a tensor T as

$$|T|_{C^{\alpha}} = \sup_{x,y} \frac{|T(x) - T(y)|}{d(x,y)^{\alpha}}$$

where the supremum is taken over all $x \neq y \in M$ that can be joined by a unique minimal geodesic, so that $|T(x) - T(y)|$ is the norm of the difference between $T(x)$ and the parallel transport of $T(y)$ to x along this minimal geodesic. Then define

$$\|T\|_{C^{k,\alpha}} := \|T\|_{C^0} + \cdots + \left\|\nabla^k T\right\|_{C^0} + \left|\nabla^k T\right|_{C^{\alpha}}.$$

Indeed, these two definitions are equivalent: they define the same space $C^{k,\alpha}(M,g)$ with uniformly equivalent norms.

Using local estimates above and a suitably chosen coordinate chart cover, we get:

Theorem 1.5 (Schauder Estimates on Compact Manifold)**.** *Let (M,g) be a compact Riemannian manifold, and L a second-order elliptic operator on M. For each $k \in \mathbb{N}$ and $\alpha \in (0,1)$, there is a constant $C = C(k,\alpha,M,g,L)$ such that if $f,g : M \to \mathbb{R}$ satisfies*

$$L(f) = g,$$

then

$$\|f\|_{C^{k+2,\alpha}(M)} \leq C\left(\|g\|_{C^{k,\alpha}(M)} + \|f\|_{L^1(M)}\right).$$

Remark 1.6. By compactness, L is uniformly elliptic once we fix a cover of M by coordinate charts. As above, C depends on the $C^{k,\alpha}$-norms of the coefficients of L, and the ellipticity constants λ, Λ. By regularity, $f \in C^{k+2,\alpha}$ if we only assume that $f \in C^2$, and the coefficients of L and g are in $C^{k,\alpha}$.

Corollary 1.7. *Let (M,g) be a compact Riemannian manifold, and L a second-order elliptic operator on M. Then*

$$\ker L := \{f \in L^2(M) \mid f \text{ is a weak solution of } Lf = 0\}$$

is a finite-dimensional space of smooth functions.

Indeed regularity theory implies that any weak solution to $Lf = 0$ is smooth.

Proof. Suppose $f_k \in \ker L$ is a sequence with $\|f_k\|_{L^2(M)} \leq 1$. By Schauder estimates above, $\|f_k\|_{C^{2,\alpha}} \leq C$ for some uniform constant C. By Arzela-Ascoli (see Theorem 1.1), f_k has a subsequence convergent in C^2, say to $f \in C^2$. Hence $f \in \ker L$ and $\|f\|_{L^2(M)} \leq 1$. We thus show that the closed unit ball in $\ker L$ is compact (in $L^2(M)$), so $\ker L$ must be finite-dimensional. ☐

We now consider the Laplacian operator on Kähler manifolds. Let (M^n, ω) be a compact Kähler manifold. Define the Laplacian with respect to metric ω in any holomorphic coordinate as

$$\Delta T := \frac{1}{2} g^{k\bar{l}} \left(\nabla_k \nabla_{\bar{l}} + \nabla_{\bar{l}} \nabla_k\right) T.$$

In particular, for functions we have

$$\Delta f = g^{k\bar{l}} \nabla_k \nabla_{\bar{l}} f = g^{k\bar{l}} \partial_k \partial_{\bar{l}} f = \mathrm{tr}_{\omega}(i\partial\bar{\partial} f).$$

Stokes' theorem shows that

$$\int_M \Delta f \omega^n = \int_M n i \partial\bar{\partial} f \wedge \omega^{n-1} = 0$$

for any smooth function f on M. Conversely, any smooth function with total mass zero is the Laplacian of some smooth function.

Theorem 1.8. *Let (M^n, ω) be a compact Kähler manifold. Suppose $\rho : M \to \mathbb{R}$ is a smooth function such that*

$$\int_M \rho\omega^n = 0,$$

then there is a smooth function $f : M \to \mathbb{R}$ such that $\Delta f = \rho$.

More generally,

Theorem 1.9 (Elliptic Operators between Hölder Spaces)**.** *Let (M, g) be a compact Riemannian manifold, and L a second-order elliptic operator with smooth coefficients on M. Fix any $k \in \mathbb{N}$ and $\alpha \in (0, 1)$. Suppose $\rho \in C^{k,\alpha}(M)$ and $\rho \perp \ker L^*$ with respect to the L^2-product. Then there exists a unique $f \in C^{k+2,\alpha}(M)$ with $f \perp \ker L$ such that $Lf = \rho$. Hence L is a Banach space isomorphism*

$$L : (\ker L)^{\perp} \cap C^{k+2,\alpha} \to (\ker L^*)^{\perp} \cap C^{k,\alpha}.$$

Compare this with Theorem 1.9 in notes on Calabi-Yau manifolds, where L is the Laplacian with respect to a Kähler metric. In particular, when $\ker L = \{0\}$ and L is self-adjoint, we get an isomorphism $L : C^{k+2,\alpha} \to C^{k,\alpha}$.

2 Chapter 4: Extremal Metrics

2.1 The Calabi Functional

Let M be a compact Kähler manifold. We are interested in finding nice Kähler metrics representing a given Kähler class $\Omega \in H^2(M,\mathbb{R})$.

Definition 2.1. An extremal metric on M in the class Ω is a critical point of the Calabi functional

$$\mathrm{Cal}(\omega) := \int_M R_\omega^2 \omega^n, \quad \omega \in \Omega \text{ Kähler metric.}$$

R_ω denotes the scalar curvature of ω.

Variational method characterize extremal metrics as the following.

Theorem 2.2. *A metric ω on M is extremal if and only if* $\mathrm{Grad}^{1,0} R_\omega$ *is a holomorphic vector field.*

Recall that we define $\mathrm{Grad}^{1,0} f$ to be $\left(\overline{\partial} f\right)^{\#}$, which is a section of $T^{1,0}M$. Similarly, $\mathrm{Grad}^{0,1} f = (\partial f)^{\#}$, so that $\mathrm{Grad}\, f = \mathrm{Grad}^{1,0} f + \mathrm{Grad}^{0,1} f = (df)^{\#}$ is the Riemannian gradient vector field of f.

Proof. Fix a Kähler metric ω and any smooth real-valued function φ. Consider the variation

$$\omega_t = \omega + ti\partial\overline{\partial}\varphi$$

inside the class $[\omega] \in H^2(M,\mathbb{R})$. Then

$$\left.\frac{d}{dt}\right|_{t=0} \omega_t^n = \Delta\varphi \cdot \omega^n,$$

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ric}(\omega_t) = -i\partial\overline{\partial}\Delta\varphi,$$

$$\left.\frac{d}{dt}\right|_{t=0} R_{\omega_t} = -\partial_p\partial_{\overline{q}}\varphi \cdot R^{p\overline{q}} - \Delta^2\varphi.$$

Using 2nd Bianchi $\nabla_{\overline{m}} R_{i\overline{j}k\overline{l}} = \nabla_{\overline{j}} R_{i\overline{m}k\overline{l}}$, hence $\nabla_{\overline{k}} R^{j\overline{k}} = g^{j\overline{k}}\nabla_{\overline{k}} R$, we get

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \mathrm{Cal}(\omega_t) &= \int_M \left(-2R(\Delta^2\varphi + R^{j\overline{k}}\partial_j\partial_{\overline{k}}\varphi) + R^2\Delta\varphi\right)\omega^n \\ &= \int_M \varphi\left(-2\Delta^2 R - 2\nabla_j\nabla_{\overline{k}}\left(R^{j\overline{k}}R\right) + \Delta(R^2)\right)\omega^n \\ &= \int_M \varphi\left(-2\Delta^2 R - 2\nabla_j\left(R^{j\overline{k}}\nabla_{\overline{k}}R\right)\right)\omega^n.\end{aligned}$$

Since φ is arbitrary, extremality of metric ω is equivalent to the condition

$$\Delta^2 R + \nabla_j\left(R^{j\overline{k}}\nabla_{\overline{k}}R\right) = 0. \tag{2.1}$$

We now try to simplify. For any function ψ, compute

$$\begin{aligned}\Delta^2\psi + \nabla_j\left(R^{j\overline{k}}\nabla_{\overline{k}}\psi\right) &= g^{j\overline{k}}g^{p\overline{q}}\nabla_j\nabla_{\overline{k}}\nabla_p\nabla_{\overline{q}}\psi + \nabla_j\left(R^{j\overline{k}}\nabla_{\overline{k}}\psi\right) \\ &= g^{j\overline{k}}g^{p\overline{q}}\nabla_j\nabla_p\nabla_{\overline{k}}\nabla_{\overline{q}}\psi - g^{j\overline{k}}g^{p\overline{q}}\nabla_j\left(\overline{R_{q\ k\overline{p}}^{m}}\nabla_{\overline{m}}\psi\right) + \nabla_j\left(R^{j\overline{k}}\nabla_{\overline{k}}\psi\right) \\ &= g^{j\overline{k}}g^{p\overline{q}}\nabla_p\nabla_j\nabla_{\overline{k}}\nabla_{\overline{q}}\psi.\end{aligned}$$

If we define $\mathcal{D} : C^\infty(M, \mathbb{C}) \to C^\infty(M, \Omega^{0,1}M \otimes \Omega^{0,1}M)$ via

$$\mathcal{D}\psi = \overline{\nabla}\overline{\nabla}\psi = \nabla_{\overline{k}}\nabla_{\overline{q}}\psi \cdot d\overline{z}^k \otimes d\overline{z}^q,$$

then Stokes' theorem yields its formal adjoint

$$\mathcal{D}^* T = g^{j\overline{k}} g^{p\overline{q}} \nabla_p \nabla_j T_{\overline{k}\overline{q}}.$$

Thus condition (2.1) is equivalent to

$$\mathcal{D}^*\mathcal{D}R = 0. \tag{2.2}$$

Integrate its product with R to see its further equivalent to

$$\mathcal{D}R = \overline{\nabla}^2 R = 0. \tag{2.3}$$

Finally, a simple calculation shows that (2.3) is equivalent to that $\mathrm{Grad}^{1,0} R = \left(\overline{\nabla}R\right)^{\#}$ be a holomorphic vector field. More generally, for a tensor $T \in C^\infty(M, \Omega^{0,1}M)$,

$$T^{\#} \text{ is holomorphic vector field} \iff \overline{\nabla}T = 0.$$

□

We call the fourth-order operator $\mathcal{D}^*\mathcal{D}$ the Lichnerowicz operator. It follows from the proof above that under the variation $\omega_t = \omega + ti\partial\overline{\partial}\varphi$, the scalar curvautre satisfies

$$\left.\frac{d}{dt}\right|_{t=0} R_{\omega_t} = -\mathcal{D}^*\mathcal{D}\varphi + \nabla_j R^{j\overline{k}} \cdot \nabla_{\overline{k}}\varphi = -\mathcal{D}^*\mathcal{D}\varphi + g^{j\overline{k}}\nabla_j R \cdot \nabla_{\overline{k}}\varphi = -\overline{\mathcal{D}^*\mathcal{D}\varphi} + g^{j\overline{k}}\nabla_j\varphi \cdot \nabla_{\overline{k}} R, \tag{2.4}$$

using that the scalar curvature is real.

One important class of examples of extremal metrics are constant scalar curvature Kähler metrics (cscK). If M does not admit non-trivial holomorphic vector fields (which is usually the case), then an extremal metric is necessarily cscK by computation above. Notice that KE metrics are cscK. Conversely, suppose ω is cscK and $c_1(M) = \lambda[\omega]$ for some $\lambda \in \mathbb{R}$ (this is necessary for ω to be KE), then ω is indeed KE. Indeed, write

$$\mathrm{Ric}_\omega = 2\pi\lambda\omega + i\partial\overline{\partial}F$$

for some real-valued smooth function F. Taking tr_ω on both sides to get

$$R = 2\pi\lambda n + \Delta F.$$

Thus ΔF is a constant, and by maximum principle, F is constant. Therefore,

$$\mathrm{Ric}_\omega = 2\pi\lambda\omega.$$

Theorem 2.3 (LeBrum-Simanca '93)**.** *Let M be a compact Kähler manifold. The set of Kähler classes admitting an extremal metric is an open subset of $H^{1,1}(M, \mathbb{R})$.*

Remark 2.4. The proof uses implicit function theorem to conclude the openness.

We now consider the relation between the Calabi functional, which is the L^2-norm of the scalar curvature, to L^2-norms of other curvature tensors. Simple calculations are involved.

Lemma 2.5. *Let α, β be real (1, 1)-forms. Suppose ω is a Kähler form, then*

$$n\alpha \wedge \omega^{n-1} = \mathrm{tr}_\omega \alpha \cdot \omega^n,$$

$$n(n-1)\alpha \wedge \beta \wedge \omega^{n-2} = \left(\mathrm{tr}_\omega \alpha \cdot \mathrm{tr}_\omega \beta - \langle \alpha, \beta \rangle_\omega\right) \omega^n.$$

Lemma 2.6. *Let M be a compact Kähler manifold. For each Kähler class Ω and each $\omega \in \Omega$,*

$$\int_M R_\omega \omega^n = 2n\pi c_1(M) \cup [\omega]^{n-1},$$

$$\int_M R_\omega^2 \omega^n = \int_M |\mathrm{Ric}_\omega|_\omega^2 \omega^n + 4\pi^2 n(n-1) c_1(M)^2 \cup [\omega]^{n-2},$$

$$\int_M |\mathrm{Ric}_\omega|_\omega^2 \omega^n = \int_M |\mathrm{Rm}_\omega|_\omega^2 \omega^n + n(n-1)\left[4\pi^2 c_1(M)^2 - 8\pi^2 c_2(M)\right] \cup [\omega]^{n-2}.$$

Proof. These are direct applications of Lemma 2.5. Recall that the Chern classes are defined via the curvature tensor

$$F := \frac{i}{2\pi} R_{i\ k\bar{l}}^{j} \left(dz^k \wedge d\bar{z}^l\right) \otimes \left(dz^i \otimes \frac{\partial}{\partial z_j}\right) \in C^\infty(M, \Lambda^{1,1}T^*M \otimes \mathrm{End}(T^{1,0}M)).$$

Then easy calculations show that

$$\mathrm{tr}\, F = \frac{1}{2\pi} \mathrm{Ric} \in c_1(M),$$

$$\begin{aligned}\left(|\mathrm{Ric}_\omega|^2 - |\mathrm{Rm}_\omega|^2\right)\omega^n &= n(n-1)\left(i^2 R_{i\ k\bar{l}}^{j} R_{j\ p\bar{q}}^{i} dz^k \wedge d\bar{z}^l \wedge dz^p \wedge d\bar{z}^q\right) \wedge \omega^{n-2} \\ &= n(n-1)\left(4\pi^2\, \mathrm{tr}\big(F^2\big)\right) \wedge \omega^{n-2} \\ &\in n(n-1)\left[4\pi^2 c_1(M)^2 - 8\pi^2 c_2(M)\right] \cup [\omega]^{n-2}.\end{aligned}$$

□

From above, we can define the average scalar curvature

$$\hat{R} := \frac{2n\pi c_1(M) \cup [\omega]^{n-1}}{[\omega]^n},$$

which is a constant depending only on the Kähler class $[\omega]$ and M. Thus writing the Calabi functional as

$$\int_M R^2 \omega^n = \int_M (R - \hat{R})^2 \omega^n + \int_M \hat{R}^2 \omega^n,$$

we can equivalently define extremal metrics as critical points of the functional

$$\omega \mapsto \int_M (R_\omega - \hat{R}_\omega)^2 \omega^n, \quad \omega \in \Omega \text{ Kähler metric.}$$

Also observe that if a cscK metric exists in a Kähler class, then it minimizes the Calabi functional over this class. More generally:

Theorem 2.7. *Let M be a compact Kähler manifold. Extremal metrics minimize the Calabi functional in their respective Kähler classes.*

2.2 The Futaki Invariant

Let (M, ω) be a compact Kähler manifold. Let $\mathfrak{h}$ denote the space of gradient holomorphic vector fields. That is, each vector field $X \in \mathfrak{h}$ can be written as

$$X = X^i \frac{\partial}{\partial z_i}$$

where X^i are holomorphic functions and $X^i = g^{j\overline{k}} \partial_{\overline{k}} f$ for some $f : M \to \mathbb{C}$. In other words,

$$\mathfrak{h} := \{ \left(\overline{\partial} f\right)^{\#} \mid f : M \to \mathbb{C} \text{ with } \overline{\nabla}^2 f = 0 \}.$$

We call f a holomorphy potential for X. Recall from §2.1 that

$$\overline{\nabla}^2 f = \mathcal{D} f = 0 \iff \mathcal{D}^* \mathcal{D} f = 0.$$

It is also useful to think of sections of $T^{1,0}M$ as real vector fields. This is achieved by defining the map $T^{1,0}M \to TM$, where TM denotes the real tangent bundle, mapping a vector field of type $(1, 0)$ to its real part. In coordinates, we describe this map over an $\mathbb{R}$-basis:

$$\frac{\partial}{\partial z_i} \mapsto \frac{1}{2} \frac{\partial}{\partial x_i}, \quad i \frac{\partial}{\partial z_i} \mapsto \frac{1}{2} \frac{\partial}{\partial y_i}.$$

Then under this map,

$$X = \left(\overline{\partial} f\right)^{\#} \mapsto \frac{1}{2} \left(\operatorname{Grad}(\operatorname{Re} f) + J \operatorname{Grad}(\operatorname{Im} f)\right),$$

where J is the complex structure and Grad is the Riemmanian gradient of a real function.

Remark 2.8. This identification $T^{1,0}M \to TM$ respects Lie bracket for holomorphic vector fields, up to a factor of 2. We shall use this fact below.

By maximum principle, each $X \in \mathfrak{h}$ has a unique holomorphy potential up to adding a constant. We can choose normalization condition that $\int_M f \omega^n = 0$, so that

$$\mathfrak{h} \cong \left\{ f : M \to \mathbb{C} \mid \overline{\nabla}^2 f = 0 \text{ and } \int_M f \omega^n = 0 \right\}.$$

Observe that $\mathfrak{h}$ is independent of the choice of metric ω in a fixed Käher class $[\omega]$:

Lemma 2.9. *Suppose $\tilde{\omega} = \omega + i\partial\overline{\partial}\varphi$ for some φ. If $X \in \mathfrak{h}$ has holomorphy potential f with respect to ω, then $f + X(\varphi)$ is a holomorphy potential for X with respect to $\tilde{\omega}$.*

Lemma 2.10. *$\mathfrak{h}$ is closed under the Lie bracket.*

Proof. Indeed the space of holomorphic vector fields is a Lie algebra. Using holomorphicity, we find

$$[\left(\overline{\partial} F\right)^{\#}, \left(\overline{\partial} G\right)^{\#}] = \left(\overline{\partial} H\right)^{\#},$$

where

$$H = \{F, G\} = g^{j\overline{k}} \left(\nabla_j G \nabla_{\overline{k}} F - \nabla_j F \nabla_{\overline{k}} G\right)$$

is the Poisson bracket of F and G. □

Next we show that $\mathfrak{h}$ is independent of the choice of Kähler class, so that $\mathfrak{h}$ is determined only by the complex structure of M. Below are two ways to see this.

Theorem 2.11 (Lebrun-Simanca)**.**

$$\begin{aligned}\mathfrak{h} &:= \{X \textit{ holomorphic vector field} \mid X = (\overline{\partial} f)^{\#} \textit{ for some smooth } f : M \to \mathbb{C}\} \\ &= \{X \textit{ holomorphic vector field} \mid X_p = 0 \textit{ for some } p \in M\} \\ &= \{X \textit{ holomorphic vector field} \mid \alpha(X) = 0 \textit{ for all holomorphic } (1,0)\textit{-forms } \alpha\}\end{aligned}$$

Proof. Let X be any holomorphic vector field. We first show that if X is gradient, then X vanishes somewhere on M. Let $f : M \to \mathbb{C}$ be a holomorphy potential for X. Then

$$X(\overline{f}) = g^{i\overline{j}} \partial_{\overline{j}} f \partial_i \overline{f} = |X|^2 = \left|\partial \overline{f}\right|^2 \geq 0.$$

Define $c = \min_M X(\overline{f}) \geq 0$, and $C = \max_M |f| \geq 0$. Let $U, V \in \mathfrak{X}(M)$ denote the real and imaginary part of X, respectively. By holomorphicity of X, we know that U, V are real holomorphic vector fields, and easily compute that $[U, V] = 0$. By compactness of M, each of U and V generates an $\mathbb{R}$-parameter group of biholomorphisms $M \to M$, and any two of these biholomorphisms commute under composition since $[U, V] = 0$. We can thus compose these biholomorphisms to get a holomorphic action $\mathbb{C} \curvearrowright M$, sending d/dz to $U + iV = X$. Let $F : \mathbb{C} \to M$ be an orbit of this action, and set $g = f \circ F$. Then $g : \mathbb{C} \to \mathbb{C}$ is a smooth function with $d\overline{g}/dz = X(\overline{f}) \circ F \geq 0$. If $D_r \subset \mathbb{C}$ is the closed disk of radius r centered at 0, Stokes' theorem yields

$$\int_{\partial D_r} g dz = \int_{D_r} \frac{dg}{d\overline{z}} d\overline{z} \wedge dz = 2i \int_{D_r} \frac{dg}{d\overline{z}} dx \wedge dy.$$

Thus

$$\frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}) e^{i\theta} d\theta = \frac{1}{2\pi i r} \int_{\partial D_r} g dz \geq \frac{c\pi r^2}{\pi r} = cr.$$

On the other hand

$$\left| \frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}) e^{i\theta} d\theta \right| \leq C = \max_M |f|.$$

Letting $r \to \infty$, we see that $c = 0$. Now we know that $X(\overline{f}) = \left|\partial \overline{f}\right|^2$ vanish at some point, hence $X = \left(\overline{\partial} f\right)^{\#}$ vanish at the same point.

Next we show that $X_p = 0$ for some $p \in M \Rightarrow \alpha(X) = 0$ for all holomorphic $(1,0)$-forms α. Note that in this case $\alpha(X) = \alpha_i X^i$ is a holomorphic function on M vanishing at p, so $\alpha(X) = 0$ by maximum principle.

Finally we show that $\alpha(X) = 0$ for all holomorphic $(1,0)$-forms $\alpha \Rightarrow X$ is gradient. Let $\varphi = X^\flat$, a $(0,1)$-form on M. Since $d\omega = 0$, an easy calculation shows that $\overline{\partial}\varphi = 0$. Let $\alpha \in \mathcal{H}^{0,1}(M, \omega)$ be an arbitrary harmonic $(0,1)$-form, and $\beta = \overline{\alpha}$. Then $\partial \alpha = 0$, so that β is holomorphic $(1,0)$-form, and

$$\langle \varphi, \alpha \rangle = \varphi_{\overline{i}} \overline{\alpha_{\overline{j}}} g^{j\overline{i}} = X^k g_{k\overline{i}} \beta_j g^{j\overline{i}} = X^k \beta_k = \beta(X) = 0.$$

Hence $\varphi \perp \mathcal{H}^{0,1}(M, \omega)$ with respect to the inner product on $\mathcal{A}^{0,1}(M)$ induced from ω. Recall that Hodge Theory gives orthogonal decomposition

$$\mathcal{A}^{0,1}(M) = \mathcal{H}^{0,1}(M, \omega) \oplus \overline{\partial} \mathcal{A}^{0,0}(M) \oplus \overline{\partial}^* \mathcal{A}^{0,2}(M).$$

Therefore, $\overline{\partial}\varphi = 0$ and $\varphi \perp \mathcal{H}^{0,1}(M, \omega)$ implies that there exists a smooth function $f : M \to \mathbb{C}$ such that $\varphi = \overline{\partial} f$. Then $X = \varphi^{\#} = \left(\overline{\partial} f\right)^{\#}$. This completes the proof. □

We have thus defined $\mathfrak{h} = \mathfrak{h}(M)$ for each compact Kähler manifold M.

Lemma 2.12. *If* $c_1(M) = 0$*, then* $\mathfrak{h} = 0$.

Proof. By Calabi-Yau, we fix a Ricci-flat Kähler metric ω. Let $X \in \mathfrak{h}$ be any holomorphic vector field with potential f. Then

$$0 = \mathcal{D}^*\mathcal{D}f = \Delta^2 f + \nabla_j \left(R^{j\overline{k}} \nabla_{\overline{k}} f \right) = \Delta^2 f.$$

Applying maximum principle twice, we see that f is constant, hence $X = 0$. ☐

Example 2.13. Let $M = \mathbb{C}^n/\Lambda$ be a complex torus. The flat metric implies $c_1(M) = 0$, so that $\mathfrak{h} = 0$. Clearly there are non-trivial holomorphic vector fields on M, e.g. $\partial/\partial z_i$. Thus $\mathfrak{h}$ in general does not contain all holomorphic vector fields.

Lemma 2.14. *If M is Fano manifold, i.e. $c_1(M) > 0$, then $\mathfrak{h}$ equals the space of all holomorphic vector fields on M.*

Proof. Recall that $c_1(M) = c_1(\mathcal{K}_M^*)$, where $\mathcal{K}_M = \Omega_M^n$ is the canonical bundle of M. Thus $\mathcal{K}_M^*$ is ample line bundle. Kodaira vanishing theorem yields

$$H^{0,1}(M) = H^1(M, \mathcal{O}_M) = H^1(M, \Omega_M^n \otimes \mathcal{K}_M^*) = 0,$$

hence $H^{1,0}(M) = H^0(M, \Omega_M) = 0$. M admits no non-trivial holomorphic $(1,0)$-forms. Proposition **??** finishes the proof. ☐

Now define Lie sub-algebra $\mathfrak{k} \subset \mathfrak{h}$ by

$$\mathfrak{k} := \{X \in \mathfrak{h} \mid X \text{ is Killing vector field under identification } T^{1,0}M = TM\}.$$

Lemma 2.15. *Let (M, ω) be a compact Kähler manifold. Suppose $X \in \mathfrak{h}$. Then $X \in \mathfrak{k}$ if and only if X has a purely imaginary holomorphy potential.*

Proposition 2.16. *Suppose ω is a cscK metric on M. Then*

$$\mathfrak{h} = \mathfrak{k} \oplus J\mathfrak{k}.$$

Proof. Recall that

$$\mathfrak{h} \cong \left\{ f : M \to \mathbb{C} \mid \mathcal{D}^*\mathcal{D}f = 0 \text{ and } \int_M f\omega^n = 0 \right\} =: \ker_0 \mathcal{D}^*\mathcal{D}.$$

Under this identification, $\mathfrak{k}$ is the subspace of $\ker_0 \mathcal{D}^*\mathcal{D}$ of purely imaginary functions by Lemma 2.15. Notice that since ω is cscK, Bianchi identity yields

$$\mathcal{D}^*\mathcal{D} = \Delta^2 + R^{j\overline{k}} \nabla_j \nabla_{\overline{k}},$$

which is a real operator. Thus $f \in \ker_0 \mathcal{D}^*\mathcal{D}$ if and only if $\operatorname{Re} f, \operatorname{Im} f \in \ker_0 \mathcal{D}^*\mathcal{D}$. This completes the proof. ☐

Example 2.17. $\mathrm{Bl}_p\mathbb{P}^2$ does not admit cscK metrics by Proposition 2.16. However, it does admit extremal metrics in every Kähler class.

More generally, Bianchi identity yields

$$\left(\mathcal{D}^*\mathcal{D} - \overline{\mathcal{D}^*\mathcal{D}} \right) \varphi = g^{j\overline{k}} \left(\nabla_j R \nabla_{\overline{k}} \varphi - \nabla_j \varphi \nabla_{\overline{k}} R \right).$$

Suppose ω is extremal such that $X_R = g^{j\overline{k}} \partial_{\overline{k}} R$ is holomorphic vector field. Let $X_f \in \mathfrak{h}$ be given by a holomorphy potential f, then

$$[X_R, X_f] = \left(\overline{\partial} \{R, f\} \right)^{\#} = g^{p\overline{q}} \nabla_{\overline{q}} g^{j\overline{k}} \left(\nabla_j f \nabla_{\overline{k}} R - \nabla_j R \nabla_{\overline{k}} f \right).$$

Let $\mathfrak{h}_R \subset \mathfrak{h}$ denote the sub-algebra commuting with X_R. Observe that $\mathfrak{k} \subset \mathfrak{h}_R$ since every Killing vector field X preserves curvature: $\mathcal{L}_X R = 0$. Then we can apply the same idea in the proof of Proposition 2.16 (splitting potential into real and imaginary parts, plus maximum principle and calculations above) to generalize it. Indeed for cscK metrics, $X_R = 0$.

Proposition 2.18. *Let ω be an extremal metric on M. Then*

$$\mathfrak{h}_R = \mathfrak{k} \oplus J\mathfrak{k}.$$

We can now define the Futaki invariant, which provides an obstruction to finding cscK metrics in a Kähler class.

Definition 2.19. Let (M, ω) be a compact Kähler manifold. Define the **Futaki invariant** $F : \mathfrak{h} \to \mathbb{C}$ by

$$F(X) := \int_M f(R - \hat{R})\omega^n, \quad X \in \mathfrak{h},$$

where f is any holomorphy potential for $X \in \mathfrak{h}$, and $\hat{R}$ is the average of the scalar curvature R.

Theorem 2.20. *The Futaki invariant is independent of the choice of metric in each fixed Kähler class. In particular, if $[\omega]$ admits a cscK metric, then $F = 0$.*

Proof. Suppose $\omega + i\partial\overline{\partial}\varphi$ is another metric in the Kähler class $[\omega]$. Then we have a family of Kähler metrics $\omega_t := \omega + ti\partial\overline{\partial}\varphi$. Let F_t denote the Futaki functional with respect to ω_t. By Lemma 2.6 and Lemma 2.9

$$F_t(X) = \int_M f_t(R_t - \hat{R})\omega_t^n,$$

where

$$f_t = f + tX(\varphi).$$

Our goal is to prove that

$$\left.\frac{d}{dt}\right|_{t=0} F_t(X) = 0.$$

Recall from (2.4) that

$$\left.\frac{d}{dt}\right|_{t=0} R_t = -\mathcal{D}^*\mathcal{D}\varphi + \nabla_j R^{j\overline{k}} \cdot \nabla_{\overline{k}}\varphi = -\mathcal{D}^*\mathcal{D}\varphi + g^{j\overline{k}}\nabla_j R \cdot \nabla_{\overline{k}}\varphi = -\overline{\mathcal{D}^*\mathcal{D}\varphi} + g^{j\overline{k}}\nabla_j\varphi \cdot \nabla_{\overline{k}}R,$$

and

$$\left.\frac{d}{dt}\right|_{t=0} \omega_t^n = \Delta\varphi \cdot \omega^n.$$

Then compute

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} F_t(X) &= \int_M X(\varphi)(R - \hat{R})\omega^n + \int_M f\left(-\overline{\mathcal{D}^*\mathcal{D}\varphi} + g^{j\overline{k}}\nabla_j\varphi \cdot \nabla_{\overline{k}}R\right)\omega^n + \int_M f(R - \hat{R})\Delta\varphi\omega^n \\
&= \int_M \left[g^{i\overline{j}}\nabla_{\overline{j}}(f\nabla_i\varphi)\right]\left(R - \hat{R}\right)\omega^n + \int_M -\mathcal{D}^*\mathcal{D}f \cdot \overline{\varphi}\omega^n + \int_M f\left(g^{j\overline{k}}\nabla_j\varphi \cdot \nabla_{\overline{k}}R\right)\omega^n \\
&= \int_M g^{i\overline{j}}\nabla_{\overline{j}}(Rf\nabla_i\varphi)\,\omega^n - \hat{R}\int_M g^{i\overline{j}}\nabla_{\overline{j}}(f\nabla_i\varphi)\,\omega^n \\
&= 0,
\end{aligned}$$

using that $\mathcal{D}f = 0$ and Stokes' theorem. □

Remark 2.21. This definition generalizes the idea in Futaki's original paper (1983), where the author considers the Kähler class $c_1(M) > 0$ of a Fano manifold. In that case we have seen in Lemma 2.14 that $\mathfrak{h}$ is the space of all holomorphic vector fields. Writing

$$\mathrm{Ric} = 2\pi\omega + i\partial\overline{\partial}F,$$

for any $\omega \in c_1(M)$, we have

$$F(X) = \int_M f(R - \hat{R})\omega^n = \int_M f\Delta F\omega^n = -\int_M X(F)\omega^n,$$

which recovers Futaki's definition.

Corollary 2.22. *Let ω be an extremal metric on a compact Kähler manifold M. If the Futaki invariant with respect to the Kähler class $[\omega]$ vanishes, then ω has constant scalar curvature.*

Proof. The scalar curvature R is the holomorphy potential of its $(1,0)$-gradient, since ω is extremal. Thus

$$0 = \int_M R(R - \hat{R})\omega^n = \int_M (R - \hat{R})^2\omega^n \Rightarrow R = \hat{R}.$$

□

2.3 The Mabuchi Functional

The Calabi functional gives a variational characterization of extremal metrics. We now focus specifically on cscK metrics, which can be characterized as critical points of the Mabuchi functional.

Let (M, ω) be a compact Kähler manifold. Define the space of Kähler potentials for the class $[\omega]$:

$$\mathcal{K} := \{\varphi \in C^\infty(M, \mathbb{R}) \mid \omega_\varphi := \omega + i\partial\overline{\partial}\varphi > 0\}.$$

From now we give lower indices φ to objects associated with the Kähler metric ω_φ.

For each $\varphi \in \mathcal{K}$, the tangent space $T_\varphi\mathcal{K}$ at φ can be identified with $C^\infty(M, \mathbb{R})$. Define a 1-form α on $\mathcal{K}$ by

$$\alpha_\varphi(\psi) = \int_M \psi \left(\hat{R} - R_\varphi\right) \omega_\varphi^n, \qquad \varphi \in \mathcal{K}, \quad \psi \in T_\varphi\mathcal{K}.$$

Remark 2.23. Consider $\mathcal{K}$ as an infinite-dimensional smooth manifold, so that the formal calculations below involving the tangent space motivate the rigorous definition of the Mabuchi functional.

Lemma 2.24. *The* 1*-form* α *on* $\mathcal{K}$ *is closed.*

Proof. We aim to show that for any $\psi_1, \psi_2 \in T_\varphi\mathcal{K}$,

$$\left.\frac{d}{dt}\right|_{t=0} \alpha_{\varphi+t\psi_2}(\psi_1) = \left.\frac{d}{dt}\right|_{t=0} \alpha_{\varphi+t\psi_1}(\psi_2),$$

so that the coefficients for $d\alpha$ in components $\psi^2 \wedge \psi^1$ and $\psi^1 \wedge \psi^2$ are equal at each $\varphi \in \mathcal{K}$, and hence $d\alpha = 0$. Compute

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \alpha_{\varphi+t\psi_2}(\psi_1) &= \int_M \left[\psi_1 \left(\mathcal{D}_\varphi^*\mathcal{D}_\varphi\psi_2 - g_\varphi^{j\overline{k}}\nabla_j R_\varphi \cdot \nabla_{\overline{k}}\psi_2\right) + \psi_1 \left(\hat{R} - R_\varphi\right) \Delta_\varphi \psi_2\right] \omega_\varphi^n \\
&= \underline{\int_M \overline{\mathcal{D}_\varphi^*\mathcal{D}_\varphi}\psi_1 \cdot \psi_2 \omega_\varphi^n} + \hat{R} \int_M \Delta_\varphi \psi_1 \cdot \psi_2 \omega_\varphi^n - \int_M \psi_2 \left(R_\varphi \Delta_\varphi \psi_1 + g_\varphi^{j\overline{k}} \nabla_j \psi_1 \cdot \nabla_{\overline{k}} R_\varphi\right) \omega_\varphi^n \\
&= \left.\frac{d}{dt}\right|_{t=0} \alpha_{\varphi+t\psi_1}(\psi_2).
\end{aligned}$$

Notice that both time derivatives are real by definition of α. ∎

Note that $\mathcal{K}$ is convex and hence contractible, so α is exact. Let $\mathcal{M} : \mathcal{K} \to \mathbb{R}$ denote the (unique) function such that $d\mathcal{M} = \alpha$ and $\mathcal{M}(0) = 0$. We call $\mathcal{M} : \mathcal{K} \to \mathbb{R}$ the **Mabuchi functional** or the **K-energy**.

We can compute $\mathcal{M}$ by integration of α. For any $\varphi \in \mathcal{K}$ and any path φ_t in $\mathcal{K}$ joining $\varphi_0 = 0$ and $\varphi_1 = \varphi$, we have

$$\frac{d}{dt}\mathcal{M}(\varphi_t) = \alpha_{\varphi_t}(\dot{\varphi}_t),$$

$$\mathcal{M}(\varphi) = \int_0^1 \frac{d}{dt}\mathcal{M}(\varphi_t) dt = \int_0^1 \alpha_{\varphi_t}(\dot{\varphi}_t) dt = \int_0^1 \int_M \dot{\varphi}_t \left(\hat{R} - R_{\varphi_t}\right) \omega_{\varphi_t}^n dt.$$

Note that $\frac{d}{dt}\mathcal{M}(\varphi_t) = 0$ if $\dot{\varphi}_t$ is a constant function on M. Therefore, by maximum principle, $\mathcal{M}$ can be identified with its descension onto the space of Kähler metrics in a fixed Kähler class $[\omega]$.

Theorem 2.25. *The critical points of the Mabuchi functional defined on the Kähler class* $[\omega]$ *are*

$$\{\varphi \in C^\infty(M, \mathbb{R}) \mid \omega_\varphi := \omega + i\partial\overline{\partial}\varphi \textit{ is cscK }\}.$$

We can equip $\mathcal{K}$ with a Riemannian metric such that $\mathcal{M}$ is convex. This provides more constraint on cscK metrics. Define the Riemannian metric on $\mathcal{K}$ by

$$\langle \psi_1, \psi_2 \rangle_\varphi = \int_M \psi_1 \psi_2 \omega_\varphi^n, \qquad \varphi \in \mathcal{K}, \quad \psi_1, \psi_2 \in T_\varphi \mathcal{K}.$$

Given this metric, let us first consider the geodesics in $\mathcal{K}$. Recall that (constant-speed) geodesics are critical points of the energy of a path with prescribed endpoints.

Proposition 2.26 (Geodesic Equation)**.** *A path φ_t in $\mathcal{K}$ is a (constant-speed) geodesic if and only if*

$$\ddot{\varphi}_t - |\partial \dot{\varphi}_t|_t^2 = \ddot{\varphi}_t - g_t^{j\bar{k}} \partial_j \dot{\varphi}_t \partial_{\bar{k}} \dot{\varphi}_t = 0.$$

Proof. Assume wlog that $t \in [0,1]$. The energy of the path φ_t with respect to this metric is

$$E(\varphi_t) = \int_0^1 \int_M \dot{\varphi}_t^2 \omega_t^n dt.$$

Then φ_t is geodesic if and only if

$$\left.\frac{d}{ds}\right|_{s=0} E(\varphi_t + s\psi_t) = 0$$

for all closed curves ψ_t with $\psi_0 = \psi_1 = 0 \in \mathcal{K}$. Compute

$$\begin{aligned} \left.\frac{d}{ds}\right|_{s=0} E(\varphi_t + s\psi_t) &= \int_0^1 \int_M \left(2\dot{\varphi}_t \dot{\psi}_t + \dot{\varphi}_t^2 \Delta_t \psi_t\right) \omega_t^n dt \\ &= \int_0^1 \int_M \left(-2\ddot{\varphi}_t \psi_t - 2\dot{\varphi}_t \psi_t \Delta_t \dot{\varphi}_t + \Delta_t(\dot{\varphi}_t^2)\psi_t\right) \omega_t^n dt \\ &= -2\int_0^1 \int_M \left(\ddot{\varphi}_t - |\partial \dot{\varphi}_t|_t^2\right) \psi_t \omega_t^n dt \end{aligned}$$

and the claim follows. □

Example 2.27. Fix a Kähler class $[\omega]$ on M. Suppose $X \in \mathfrak{h}$ has **real-valued** holomorphy potential $u : M \to \mathbb{R}$. Let $X_\mathbb{R}$ denote the real part of X, i.e. the image of X under the identification $T^{1,0}M = TM$ discussed in §2.2. We know already that

$$X_\mathbb{R} = \frac{1}{2} \operatorname{Grad} u,$$

and $X_\mathbb{R}$ is a real holomorphic vector field, i.e. the 1-parameter group of diffeomorphisms $f_t : M \to M$ generated by $X_\mathbb{R}$ preserves the complex structure $J \in C^\infty(M, \operatorname{End}(TM))$ of M:

$$\mathcal{L}_{X_\mathbb{R}} J = 0.$$

This is easily seen using Cauchy-Riemann equations for holomorphic functions. Now that f_t^* preserves the type, we are able to define the path of metrics

$$\omega_t := f_t^* \omega.$$

It is easy to verify that (take time derivative of ω_t and integrate)

$$\omega_t = \omega + i\partial\bar{\partial}\varphi_t,$$

where

$$\dot{\varphi}_t = f_t^* u, \quad \varphi_0 = 0.$$

Then φ_t is a geodesic line in $\mathcal{K}$. For example, at $t = 0$,

$$\ddot{\varphi}_0 = \mathcal{L}_{X_{\mathbb{R}}} u = |\partial u|^2_\omega = |\partial \dot{\varphi}_0|^2_0.$$

The derivative of the Mabuchi functional along this geodesic is

$$\begin{aligned}
\frac{d}{dt}\mathcal{M}(\varphi_t) &= \int_M \dot{\varphi}_t \left(\hat{R} - R_t\right) \omega_t^n \\
&= \int_M f_t^* \left[u\left(\hat{R} - R\right)\omega^n\right] \\
&= \int_M u\left(\hat{R} - R\right)\omega^n \\
&= -F(X),
\end{aligned}$$

where $F : \mathfrak{h} \to \mathbb{C}$ is the Futaki invariant with respect to $[\omega]$. Thus the Mabuchi functional is linear along this geodesic φ_t. More generally,

Proposition 2.28. *The Mabuchi functional $\mathcal{M} : \mathcal{K} \to \mathbb{R}$ is convex along geodesics.*

Proof. Let φ_t be any geodesic. Compute

$$\frac{d}{dt}\mathcal{M}(\varphi_t) = \int_M \dot{\varphi}_t \left(\hat{R} - R_t\right) \omega_t^n,$$

using Proposition 2.26,

$$\begin{aligned}
\frac{d^2}{dt^2}\mathcal{M}(\varphi_t) &= \int_M \left[\ddot{\varphi}_t\left(\hat{R} - R_t\right) + \dot{\varphi}_t\left(\mathcal{D}_t^*\mathcal{D}_t\dot{\varphi}_t - g_t^{j\overline{k}}\nabla_j R_t \nabla_{\overline{k}}\dot{\varphi}_t\right) + \dot{\varphi}_t\left(\hat{R} - R_t\right)\Delta_t\dot{\varphi}_t\right]\omega_t^n \\
&= \int_M |\mathcal{D}_t\dot{\varphi}_t|_t^2\omega_t^n + \int_M g_t^{j\overline{k}}\nabla_j\left(\dot{\varphi}_t\left(\hat{R} - R_t\right)\nabla_{\overline{k}}\dot{\varphi}_t\right)\omega_t^n \\
&= \int_M |\mathcal{D}_t\dot{\varphi}_t|_t^2\omega_t^n \\
&\geq 0.
\end{aligned}$$

□

Example 2.29. Suppose $\omega_0, \omega_1 = \omega_0 + i\partial\overline{\partial}\varphi$ are two cscK metrics in the same Kähler class on M, and there is a geodesic path φ_t connecting $\varphi_0 = 0$ and $\varphi_1 = \varphi$ in $\mathcal{K}$. Then we claim that there is a biholomorphism $f : M \to M$ such that $f^*\omega_1 = \omega_0$. To see this, note first that

$$\frac{d}{dt}\mathcal{M}(\varphi_t) = 0 \quad \text{for } t = 0, 1,$$

as ω_0 and ω_1 are cscK. By Proposition 2.28, we see that $\mathcal{M}(\varphi_t)$ is constant for $t \in [0, 1]$, and

$$\mathcal{D}_t\dot{\varphi}_t = 0, \quad t \in [0, 1].$$

Thus $\dot{\varphi}_t$ is a real-valued holomorphy potential for its gradient vector field $X_t = g_t^{j\overline{k}}\partial_{\overline{k}}\dot{\varphi}_t \in \mathfrak{h}$ for all t. As in Example 2.27, the real part of each X_t,

$$X_{t,\mathbb{R}} = \frac{1}{2}\operatorname{Grad}_t \dot{\varphi}_t$$

is a real holomorphic vector field. Let f_t be the family of biholomorphisms generated by $-X_{t,\mathbb{R}}$, that is,

$$f_t : M \to M, \quad \frac{d}{dt}f_t(p) = -X_{t,\mathbb{R}}(f_t(p)).$$

Then $f_t^*\omega_t \equiv \omega_0$ is constant Kähler metric, and hence f_1 is the desired biholomorphism, because

$$\mathcal{L}_{X_{t,\mathbb{R}}}\omega_t = i\partial\overline{\partial}\dot{\varphi}_t,$$

$$\frac{d}{dt}\left(f_t^*\omega_t\right) = f_t^*\left(\mathcal{L}_{-X_{t,\mathbb{R}}}\omega_t + i\partial\overline{\partial}\dot{\varphi}_t\right) = 0.$$

2.4 The Ding Functional

The Ding functional gives a variational characterization of KE metrics when M is Fano, i.e. $c_1(M) > 0$. The notations are ideas in this subsection are similar to the previous one.

Now suppose M is Fano and fix $\omega \in c_1(M)$. For each Kähler potential $\varphi \in \mathcal{K}$ define the Ricci potential $h_\varphi \in C^\infty(M, \mathbb{R})$ to be the unique function such that

$$\mathrm{Ric}(\omega_\varphi) = \omega_\varphi + i\partial\overline{\partial}h_\varphi \tag{2.5}$$

with normalization

$$\int_M e^{h_\varphi}\omega_\varphi^n = \int_M \omega_\varphi^n. \tag{2.6}$$

Lemma 2.30. *Let φ_t be any curve in $\mathcal{K}$. Then*

$$\dot{h}_t + \dot{\varphi}_t + \Delta_t\dot{\varphi}_t = C_t$$

for some constant function C_t on M for each t. Also,

$$C_t \int_M \omega_t^n = \int_M e^{h_t}\dot{\varphi}_t\omega_t^n.$$

Proof. Take time derivatives of (2.5) and (2.6). □

Lemma 2.31. *Define 1-form α on $\mathcal{K}$ via*

$$\alpha_\varphi(\psi) = \int_M \psi\left(e^{h_\varphi} - 1\right)\omega_\varphi^n.$$

Then α is closed, and hence exact.

Proof. Use the lemma above and the idea in §2.3. We eventually get

$$\left.\frac{d}{dt}\right|_{t=0}\alpha_{\varphi+t\psi_2}(\psi_1) = \int_M \left(-\psi_1\Delta_\varphi\psi_2 - \psi_1\psi_2e^{h_\varphi}\right)\omega_\varphi^n + \left(\int_M \omega_\varphi^n\right)^{-1}\left(\int_M \psi_1 e^{h_\varphi}\omega_\varphi^n\right)\left(\int_M \psi_2 e^{h_\varphi}\omega_\varphi^n\right).$$

□

We can thus define the Ding functional $\mathcal{F} : \mathcal{K} \to \mathbb{R}$ such that $d\mathcal{F} = \alpha$. The variation along any path is

$$\frac{d}{dt}\mathcal{F}(\varphi_t) = \alpha_{\varphi_t}(\dot{\varphi}_t) = \int_M \dot{\varphi}_t\left(e^{h_t} - 1\right)\omega_t^n. \tag{2.7}$$

Clearly, the critical points of $\mathcal{F}$ are Kähler potentials φ such that $h_\varphi = 0$, which induce KE metrics.

Lemma 2.32. *If $f \in C^\infty(M, \mathbb{R})$ such that*

$$\int_M fe^{h_\varphi}\omega_\varphi^n = 0,$$

then

$$\int_M f^2e^{h_\varphi}\omega_\varphi^n \le \int_M |\partial f|_\varphi^2 e^{h_\varphi}\omega_\varphi^n.$$

Proof. For details consult Futaki's paper, *Kähler Einstein Metrics and Integral Invariants*, p.40. The essential idea is to observe that the operator

$$L_\varphi : C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C}),$$

$$L_\varphi u = -\Delta_\omega u - g^{i\bar{j}} \nabla_i \varphi \nabla_{\bar{j}} u = -\Delta_\omega u - \langle \nabla\varphi, \nabla\overline{u} \rangle$$

is elliptic and self-adjoint with respect to the weighted inner product

$$(u, v)_\varphi = \int_M u\overline{v} e^\varphi \omega_\varphi^n,$$

and

$$(L_\varphi u, u)_\varphi = \left(\overline{\nabla} u, \overline{\nabla} u\right)_\varphi \geq 0.$$

Hence L_φ has eigenvalues $0 = \lambda_0 < \lambda_1 \leq \dots$, tending to ∞, and calculation shows that $\lambda_1 \geq 1$. The claim now follows by considering $(L_\varphi f, f)_\varphi$. □

Theorem 2.33. *The Ding functional $\mathcal{F} : \mathcal{K} \to \mathbb{R}$ is convex along smooth geodesics in $\mathcal{K}$.*

Proof. Let φ_t be a geodesic in $\mathcal{K}$. Using Proposition 2.26, Lemma 2.30, and (2.7), compute

$$\begin{aligned}
\frac{d^2}{dt^2}\mathcal{F}(\varphi_t) &= \int_M e^{h_t} \left(\ddot{\varphi}_t + \dot{\varphi}_t \dot{h}_t + \dot{\varphi}_t \Delta_t \dot{\varphi}_t \right) \omega_t^n - \int_M \left(\ddot{\varphi}_t + \dot{\varphi}_t \Delta_t \dot{\varphi}_t \right) \omega_t^n \\
&= \int_M e^{h_t} \left(|\partial \dot{\varphi}_t|_t^2 + \dot{\varphi}_t \dot{h}_t + \dot{\varphi}_t \Delta_t \dot{\varphi}_t \right) \omega_t^n \\
&= \int_M |\partial \dot{\varphi}_t|_t^2 e^{h_t} \omega_t^n + \int_M e^{h_t} \dot{\varphi}_t \left(C_t - \dot{\varphi}_t \right) \omega_t^n \\
&= \int_M \left(|\partial \dot{\varphi}_t|_t^2 - \dot{\varphi}_t^2 \right) e^{h_t} \omega_t^n + C_t^2 \int_M \omega_t^n \\
&\geq 0.
\end{aligned}$$

□

Compared to the Mabuchi functional, the Ding functional can be defined for metrics with less regularity, and the convexity of $\mathcal{F}$ along geodesics still holds in that case.